

# AUTOMATIC CONTINUITY OF ABSTRACT HOMOMORPHISMS BETWEEN LOCALLY COMPACT AND POLISH GROUPS

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We are concerned with questions of the following type. Suppose that  $G$  and  $K$  are topological groups belonging to a certain class  $\mathcal{K}$  of spaces, and suppose that

$$K \xrightarrow{\varphi} G$$

is an abstract (i.e. not necessarily continuous) surjective group homomorphism. Under what conditions on the group  $G$  and the kernel  $\ker(\varphi)$  is the homomorphism  $\varphi$  automatically continuous and open? Questions of this type have a long history and were studied in particular for the case that  $G$  and  $K$  are Lie groups, compact groups, or Polish groups.

We develop an axiomatic approach, which allows us to resolve the question uniformly for different classes of topological groups. In this way we are able to extend the classical results about automatic continuity to a much more general setting. We shall say that a class  $\mathcal{K}$  of topological Hausdorff spaces which is closed under the passage to closed subspaces and closed under finite products is *almost Polish* if every space  $X$  in the class satisfies the following properties:

- (1) Every open covering of  $X$  has a countable subcovering.
- (2) The space  $X$  is not a countable union of nowhere dense subsets.
- (3) For each continuous image  $A \subseteq X$  of some  $\mathcal{K}$ -member there is an open set  $U \subseteq X$  such that the symmetric difference  $(A - U) \cup (U - A)$  is a countable union of nowhere dense subsets of  $X$ .

Note that by (3), being almost Polish is a property of a class of spaces, and not a property of an individual space. The class  $\mathcal{P}$  of Polish spaces, the class  $\mathcal{L}^\sigma$  of locally compact  $\sigma$ -compact spaces and the class  $\mathcal{C}$  of compact spaces are almost Polish. A more systematic discussion of almost Polish classes of spaces will be presented in Section 2 below.

Given an almost Polish class  $\mathcal{K}$  and a space  $X \in \mathcal{K}$ , we call a subset  $A \subseteq X$  a  $\mathcal{K}$ -*analytic set* if  $A = \psi(Z)$  holds for some  $Z \in \mathcal{K}$  and some continuous map  $\psi : Z \rightarrow X$ . This notion parallels the notion of an analytic set (or Suslin set) in the classical theory of Polish spaces and in descriptive set theory. Singletons are  $\mathcal{K}$ -analytic in every almost Polish class  $\mathcal{K}$ .

Suppose that  $\mathcal{K}$  is an almost Polish class of spaces, and that  $G$  is a topological group that belongs to  $\mathcal{K}$ . We now introduce a piece of terminology which we shall use throughout

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this text. We shall call  $G$  *rigid within the class  $\mathcal{K}$* , or  $\mathcal{K}$ -*rigid* for short, if the following holds.

**Rigidity.** *For every short exact sequence of groups*

$$1 \longrightarrow N \hookrightarrow K \xrightarrow{\varphi} G \longrightarrow 1,$$

*where  $\varphi$  is an abstract group homomorphism, where  $G$  and  $K$  belong to  $\mathcal{K}$ , and where the kernel  $N$  of  $\varphi$  is  $\mathcal{K}$ -analytic, the homomorphism  $\varphi$  is automatically continuous and open.*

Notice right away that any topological group  $G$  supporting a discontinuous automorphism fails to be  $\mathcal{K}$ -rigid for any almost Polish class  $\mathcal{K}$  containing  $G$ . The following consequence of rigidity is immediate from the definition.

**Uniqueness of group topologies.** *A  $\mathcal{K}$ -rigid topological group  $G$  for an almost Polish class  $\mathcal{K}$  has a unique group topology in that class  $\mathcal{K}$ . In particular, every abstract group automorphism of  $G$  is a homeomorphism.*

Before we formulate our first rigidity result we need to recall that a simple real Lie algebra  $\mathfrak{g}$  is called *absolutely simple* if its complexification  $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$  is a simple complex Lie algebra. The simple real Lie algebras which are not absolutely simple are precisely the complex simple Lie algebras, viewed as real Lie algebras.

**Theorem A.** *A Lie group  $G$  is rigid within every almost Polish class  $\mathcal{K}$  containing it, provided it satisfies the following conditions:*

- (1) *The center  $\text{Cen}(G^\circ)$  of its identity component is finite.*
- (2) *Its Lie algebra  $\text{Lie}(G)$  is a direct sum of absolutely simple ideals.*

See Theorem 4.6 for the proof. Theorem A generalizes results in [2], [5], [7], [41], and [42], which mainly concern automatic continuity of abstract isomorphisms within the class of semisimple Lie groups, and [21, 5.66], [24], [25], and [28] which concern rigidity with respect to subclasses of  $\mathcal{P}$  and  $\mathcal{L}^\sigma$ . We also prove rigidity results for certain semidirect products of vector groups and classical Lie groups in Section 5. We refer to Theorem 5.5 and Theorem 5.6.

Following [21, 9.5], we shall call a compact topological group  $G$  *semisimple* if it is connected and perfect.

**Theorem B.** *A compact semisimple group  $G$  is rigid within every almost Polish class  $\mathcal{K}$  containing it.*

In Theorem 7.7 below we prove in fact a rigidity result which holds for a much larger class of compact groups than semisimple groups, including many profinite groups. For the class  $\mathcal{L}^\sigma$ , Theorem B is essentially proved in [4], and in a more restricted form in [39].

In a different direction, and building on work by Nikolov–Segal, we obtain the following result, which generalizes [9]. See Theorem 6.3.

**Theorem C.** *A topologically finitely generated profinite group  $G$  is rigid within every almost Polish class  $\mathcal{K}$  containing it.*

All these rigidity results deal with abstract homomorphisms  $K \longrightarrow G$  where the *range*  $G$  has prescribed properties. However, the methods that we develop are also capable of producing automatic continuity results for abstract homomorphisms  $G \longrightarrow H$  between topological groups where the *domain*  $G$  is a Lie group with special properties.

For this purpose, we call a subset  $C$  of a topological group *spacious* if some product of finitely many left translates of  $CC^{-1}$  has nonempty interior. Small spacious sets abound in many Lie groups, as we shall show in Section 3. Theorem D generalizes [21, 5.64] and [42], see Theorem 8.1 and its corollaries.

**Theorem D.** *Let  $\psi : G \longrightarrow H$  be an abstract homomorphism from a Lie group  $G$  into a topological group  $H$  satisfying the following conditions:*

- (1) *The Lie algebra  $\text{Lie}(G)$  is perfect.*
- (2) *There exists a compact spacious set  $C \subseteq G$  whose image  $\psi(C) \subseteq H$  has compact closure.*

*Then  $\psi$  is continuous.*

We also take the opportunity to correct a mistake which occurred both in [28] and in [4], see Section 9 below.

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**An outline of the main argument.** Suppose that we are given a short exact sequence

$$1 \longrightarrow N \hookrightarrow K \xrightarrow{\varphi} G \longrightarrow 1,$$

where  $G, K$  are groups in some almost Polish class  $\mathcal{K}$ , where  $\varphi$  is an abstract group homomorphism, and where  $N \trianglelefteq K$  is a normal (not necessarily closed) subgroup which is a  $\mathcal{K}$ -analytic set. As a first step, we show that  $\varphi$  is continuous and open, provided that we can construct a neighborhood basis of the identity in  $G$  consisting of sets  $V \subseteq G$  whose preimages  $\varphi^{-1}(V)$  are  $\mathcal{K}$ -analytic. This is the main result in Section 2.

The actual construction of this neighborhood basis depends on the nature of the group  $G$ . If  $G$  is an  $n$ -dimensional Lie group whose Lie algebra is perfect, then we show the following. If  $D \subseteq G$  is one fixed compact identity neighborhood, then there exist 1-parameter subgroups  $c_1, \dots, c_n$  in  $G$  such that the family  $M_{t_1, \dots, t_n} = [c_1(t_1), D] \cdots [c_n(t_n), D]$ , where  $0 < t_i \leq 1$  holds for all  $i = 1, \dots, n$ , is a neighborhood basis of the identity in  $G$ . This observation goes back to van der Waerden.

Suppose that  $G$  is a compact connected semisimple Lie group. In this case we may put  $D = G$  and then apply van der Waerden's construction. It turns out that then the sets  $\varphi^{-1}(M_{t_1, \dots, t_n})$  are  $\mathcal{K}$ -analytic. In view of the results in Section 2, this allows us to conclude that  $\varphi$  is continuous and open.

If  $G$  is a connected semisimple Lie group with finite center, but not compact, then harder work is required. In this case we show that there exists a finite set  $X \subseteq G$  and an

element  $h \in G$  such that the set  $C = \{ghg^{-1} \mid g \in \text{Cen}_G(X)\}$  is compact, and such that there exist elements  $g_1, \dots, g_r \in G$  such that  $D = g_1CC^{-1}g_2CC^{-1} \dots g_rCC^{-1} \subseteq G$  is a compact identity neighborhood. This result depends on the advanced structure theory of real semisimple Lie groups. It requires that each simple ideal in the Lie algebra of  $G$  is absolutely simple. We also use Yamabe's Theorem saying that a path connected subgroup of a Lie group is an analytic Lie subgroup, and Baire's Category Theorem. In any case, it follows again that the sets  $\varphi^{-1}(M_{t_1, \dots, t_n})$  are  $\mathcal{K}$ -analytic in  $K$ , and by the results in Section 2, the map  $\varphi$  is continuous and open.

For general compact connected semisimple groups and for topologically finitely generated profinite groups, the arguments are somewhat different. However, they always boil down to the construction of a neighborhood basis of the identity in  $G$  consisting of sets that arise in an 'algebraic' way starting from a finite set of group elements.

## 1. EXAMPLES AND COUNTEREXAMPLES

Before we embark on the proofs of our main results, we collect a series of examples which illustrate that our main results fail if certain assumptions are dropped.

**1.1. Rigidity fails for abelian groups.** As abstract groups,  $\mathbb{C}^*$  and  $U(1) \cong \mathbb{R}/\mathbb{Z}$  are isomorphic [21, A1.43]. Hence uniqueness of topologies fails for these locally compact Polish abelian groups. Also, the compact group  $\mathbb{R}/\mathbb{Z}$  has non-continuous abstract automorphisms.

**1.2. Rigidity fails for groups which are not locally compact or  $\sigma$ -compact.** By [27], the field  $\mathbb{R}$  admits uncountably many nondiscrete field topologies, which are not locally compact. Let  $\mathcal{T}$  be one such topology which is different from the usual topology  $\mathcal{S}$  on the reals. Then the matrix group  $G = \text{SO}(3) \subseteq \mathbb{R}^{3 \times 3}$  becomes a topological group with respect to  $\mathcal{T}$ , but the identity map  $(\text{SO}(3), \mathcal{T}) \rightarrow (\text{SO}(3), \mathcal{S})$  is not continuous. Such a topology on  $G$  fails to be locally compact or Polish.

The discrete topology  $\mathcal{D}$  on  $\text{SO}(3)$  is locally compact and metrizable, but neither  $\sigma$ -compact nor Polish. The identity map  $(\text{SO}(3), \mathcal{D}) \rightarrow (\text{SO}(3), \mathcal{S})$  is a continuous bijective homomorphism from the discrete group to the compact matrix group, which is not open.

**1.3. Rigidity fails for infinite products of compact Lie groups if the kernel is not restricted.** (See [13, pp. 182–183].) Let  $I$  be an infinite set, let  $G$  be a compact group and let

$$K = \prod_{i \in I} G = \text{Map}(I, G).$$

Let  $I \hookrightarrow \beta I$  denote the Čech–Stone compactification of the discrete space  $I$ . For  $x \in \beta I$  put  $\mu(x) = \{J \subseteq I \mid x \in \overline{J}\}$ . Then  $\mu(x)$  is an ultrafilter on  $I$  which is free if and only if

$x \in \beta I - I$ . From the universal property of the Čech–Stone compactification we obtain a bijection

$$\text{Map}(I, G) \hookrightarrow C(\beta I, G).$$

For  $x \in \beta I$ , the evaluation homomorphism  $x^* : C(\beta I, G) \rightarrow G$ ,  $f \mapsto f(x)$  is surjective, because  $G$  embeds diagonally in  $\text{Map}(I, G) \subseteq C(\beta I, G)$  as the set of constant maps. If  $x = j \in I$ , then  $x^* = \text{pr}_j$  is the projection onto the  $j$ th coordinate, which is continuous and open. However if  $x \in \beta I - I$ , then the kernel of  $x^*$  is dense in  $K$ , as can be seen from the fact that then the ultrafilter  $\mu(x)$  contains all cofinite sets.<sup>1</sup>

Suppose that  $I$  is countably infinite and that  $G = \text{Alt}(5)$  or that  $G = \text{SO}(3)$ . Then  $G$  and  $K$  are compact Polish groups and  $x^* : K \rightarrow G$  is surjective, but not continuous if  $x \in \beta I - I$ . This shows that we have to make some topological assumption on the kernel of the homomorphism in order to obtain rigidity results.

In view of Theorem B and its generalization, Theorem 7.7, we see also that  $\ker(x^*)$  is then neither  $\sigma$ -compact nor analytic. In the case that  $G = \text{Alt}(5)$  we can even conclude that the finite index subgroup  $\ker(x^*) \subseteq K$  is not Haar measurable, since otherwise  $\ker(x^*)$  would necessarily have positive and finite volume, and hence would be open and closed.

#### 1.4. Rigidity fails for $\text{SL}_n(\mathbb{C})$ and all infinite complex linear algebraic groups.

The field of complex numbers has  $2^{2^{\aleph_0}}$  non-continuous automorphisms. Each of these automorphisms extends entry-wise to a non-continuous automorphism of the matrix group  $\text{SL}_n(\mathbb{C})$ . More generally, this method gives non-continuous automorphisms of all infinite complex matrix groups  $\mathbf{G}(\mathbb{C})$ , where  $\mathbf{G}$  is a linear algebraic group defined over  $\mathbb{C}$ . Such a group  $\mathbf{G}(\mathbb{C})$  is in a natural way a complex Lie group.

But there are even more topologies on the field  $\mathbb{C}$ . Let  $p$  be a prime and let  $\mathbb{C}_p$  denote the completion of the algebraic closure of the field of  $p$ -adic numbers. Then  $\mathbb{C}_p$  is an algebraically closed separable complete valued field, and thus  $\text{SL}_n(\mathbb{C}_p)$  is a Polish group. Since  $\mathbb{C}_p$  is algebraically closed, of characteristic 0 and of cardinality  $2^{\aleph_0}$ , there is a field isomorphism  $\mathbb{C} \cong \mathbb{C}_p$ . This shows that  $\text{SL}_n(\mathbb{C}) \cong \text{SL}_n(\mathbb{C}_p)$  carries many non-homeomorphic Polish group topologies. These complex Lie groups typically have a simple Lie algebra which fails to be absolutely simple.

The field of real numbers has only one automorphism, so this construction does not carry over. The next example, which is due to J. Tits, shows that real algebraic groups may nevertheless have non-continuous automorphisms.

**1.5. Rigidity fails for certain connected perfect real algebraic groups.** Let  $\mathbf{G}$  be a linear algebraic group defined over  $\mathbb{R}$ , e.g.  $\mathbf{G} = \text{SL}_n$ . Then  $G = \mathbf{G}(\mathbb{R})$  is a real Lie group. Let  $\mathbb{R}[\delta] = \mathbb{R}[x]/(x^2)$  denote the ring of dual numbers. The tangent bundle group  $TG$  on the one hand is isomorphic to the semidirect product  $\text{Lie}(G) \rtimes_{\text{Ad}} G$ , and on the other hand is isomorphic to the group of  $\mathbb{R}[\delta]$ -points  $\mathbf{G}(\mathbb{R}[\delta])$ . It is observed in [41] that

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<sup>1</sup> From a different viewpoint, we express the group  $G$  here via  $x^*$  as an *ultralimit* or *asymptotic cone* of a constant family  $\{G\}$  of compact metric groups, without rescaling.

the ring  $\mathbb{R}[\delta]$  has uncountably many non-continuous ring automorphisms, which extend functorially to non-continuous automorphisms of the Lie group  $\mathbf{G}(\mathbb{R}[\delta])$ . For  $\mathbf{G} = \mathrm{SO}_n$  ( $n \geq 3$ ) or  $\mathbf{G} = \mathrm{SL}_n$  ( $n \geq 2$ ), the Lie group  $\mathbf{G}(\mathbb{R}[\delta])$  is a connected perfect Lie group, see Section 5 below. These Lie groups  $\mathbf{G}(\mathbb{R}[\delta])$  are therefore not rigid within  $\mathcal{L}^\sigma$  and  $\mathcal{P}$ . A particular case of this phenomenon is the group

$$G = \mathbb{R}^3 \rtimes_\rho \mathrm{SO}(3),$$

where  $\rho : \mathrm{SO}(3) \hookrightarrow \mathrm{GL}_3(\mathbb{R})$  is the standard representation. In this case  $\rho = \mathrm{Ad}$  is the adjoint representation, and the resulting group  $G = \mathrm{SO}_3(\mathbb{R}[\delta])$  is not rigid.

## 2. BACKGROUND ON TOPOLOGICAL GROUPS AND POINT-SET TOPOLOGY

Our convention is that all topological groups and spaces are Hausdorff. All other topological conditions will be stated explicitly. A *path* in a space  $X$  is a continuous map  $f : [0, 1] \rightarrow X$ . If there exists a nonconstant path in  $X$  then we say that  $X$  *contains a nonconstant path*.

A set will be called *countable* if its cardinality does not exceed  $\aleph_0$ .

The identity component of a topological group  $G$  is denoted by  $G^\circ$ . This is a closed normal subgroup, and  $G/G^\circ$  is a totally disconnected topological group [14, II.7.1 and II.7.3]. The topological group  $G$  is called *almost connected* if  $G/G^\circ$  is compact.

Recall that a space is called  *$\sigma$ -compact* if it can be written as a countable union of compact sets. Since locally compact  $\sigma$ -compact groups play a prominent role in this article, it seems worthwhile to take a closer look at their structure. Theorem 2.1 below will not be used elsewhere, but it may well take the magic out of this class of groups.

**Theorem 2.1.** *For a locally compact group  $G$  the following are equivalent.*

- (1)  *$G$  is  $\sigma$ -compact.*
- (2) *Every open subgroup  $H \subseteq G$  has countable index.*
- (3)  *$G$  has an almost connected open subgroup of countable index.*

*Proof.* If  $G$  is  $\sigma$ -compact and  $H \subseteq G$  is open, then  $G/H$  is discrete and  $\sigma$ -compact, and thus countable. Hence (1) implies (2). By [31, Lemma 2.3.1], (2) implies (3).

Going from (3) to (1) is much deeper. Suppose that  $H \subseteq G$  is an almost connected open subgroup of countable index. We must show that  $G$  is  $\sigma$ -compact. By Iwasawa's Splitting Theorem, see [23, p. 547, Theorem 1], [10, Theorem A] for the local version and [18, Theorem 4.1], [17, Theorem 4.4] for the global version, there exists a compact subgroup  $K \subseteq H$ , a 1-connected Lie group  $L$  and an open continuous homomorphism  $\varphi : L \times K \rightarrow H$  with discrete kernel, and with  $\varphi(1, k) = k$ . In particular, the group  $H' = \varphi(L \times K)$  is open in  $H$ . Every connected locally compact group is compactly generated and thus  $\sigma$ -compact. Therefore  $L$  is  $\sigma$ -compact, and hence  $H'$  is  $\sigma$ -compact. The homomorphism  $\pi : H \rightarrow H/H^\circ$  annihilates the connected group  $\varphi(L \times \{1\})$ , hence  $\pi(H') = \pi(\varphi(\{1\} \times K))$  is compact and open in  $H/H^\circ$ . Thus  $\pi(H')$  has finite index in  $H/H^\circ$ . Since  $H' \subseteq H$  is open, we have that  $H^\circ \subseteq H'$  and thus  $H'$  has finite index in

$H$ . In particular, the  $\sigma$ -compact group  $H'$  has countable index in  $G$ . Then  $G$ , being a countable union of  $\sigma$ -compact sets, is  $\sigma$ -compact.

An alternative but equally nontrivial proof of a more geometric nature may be instructive. Let  $H$  be an open almost connected subgroup of countable index in  $G$ . We have to show that  $H$  is  $\sigma$ -compact, for then  $G$  will be  $\sigma$ -compact as well. Now  $H$  contains a subspace  $E$  homeomorphic to  $\mathbb{R}^n$  for some  $n \in \{0, 1, 2, \dots\}$  and a (maximal) compact subgroup  $C$  such that  $(e, c) \mapsto ec : E \times C \rightarrow H$  is a homeomorphism. For a proof that is even valid for almost connected pro-Lie groups see [20, Corollary 8.5 on p. 380], and also [22]. Since  $\mathbb{R}^n \times C$  is clearly  $\sigma$ -compact, so is  $H$ .  $\square$

In order to prove continuity and openness of abstract group homomorphisms, we will use Pettis' Theorem. This requires several notions related to Baire's Category Theorem, which we review now.

**2.2. Meager sets, Baire spaces and almost open sets.** A subset  $A$  of a space  $X$  is called *nowhere dense* if its closure  $\overline{A}$  has empty interior. A subset  $B \subseteq X$  which is contained in a countable union of nowhere dense sets is called *meager* (or of *first category* in older books). A countable union of meager sets is again meager. If every meager subset of  $X$  has empty interior, then  $X$  is called a *Baire space*. In a Baire space, Baire's Category Theorem is valid [6, XI.10].

A subset  $E \subseteq X$  of a space  $X$  is called *almost open* if there exists an open set  $U \subseteq X$  such that the symmetric difference  $(E - U) \cup (U - E)$  is meager. Almost open sets are also called *Baire measurable*. The almost open sets form a  $\sigma$ -algebra [34, Theorem 4.3], which contains all Borel sets in  $X$ .

**2.3. Polish spaces and analytic sets.** A space is called *Polish* if it is second countable and metrizable by a complete metric. A subset  $A$  of a Polish space  $X$  is called *analytic* or *Suslin* if there exists a Polish space  $Z$  and an continuous map  $\psi : Z \rightarrow X$  with  $A = \psi(Z)$ . Analytic sets have remarkable properties; among other things, they are almost open. On the other hand, the class of Polish spaces is way too narrow for our purposes. We thus propose in 2.7 a formal definition which captures those properties of Polish spaces and analytic sets which are important for us. We refer to [3], [26], and [29] for more results on Polish spaces and analytic sets.

**2.4.  $\mathcal{K}$ -analytic sets.** Let  $\mathcal{K}$  be a class of topological spaces which is closed under finite products and under passage to closed subsets. In other words, if  $X, Y \in \mathcal{K}$  and if  $A \subseteq X$  is closed, then  $A, X \times Y \in \mathcal{K}$ . Examples of classes satisfying these assumptions are the class  $\mathcal{P}$  of Polish spaces, the class  $\mathcal{C}$  of compact spaces, and the class  $\mathcal{L}^\sigma$  of locally compact  $\sigma$ -compact spaces.

Given such a class  $\mathcal{K}$  and a space  $X$  in  $\mathcal{K}$ , we call a subset  $A \subseteq X$  a  *$\mathcal{K}$ -analytic set* if there exists a space  $Z \in \mathcal{K}$  and a continuous map  $\psi : Z \rightarrow X$  with  $\psi(Z) = A$ . We

let  $\mathcal{K}_a$  denote the class of all pairs  $(X, A)$ , where  $X \in \mathcal{K}$  and  $A \subseteq X$  is  $\mathcal{K}$ -analytic. The following is straightforward.

**Lemma 2.5.** *If  $X \in \mathcal{K}$  and  $(Y, B) \in \mathcal{K}_a$  and if  $\varphi : X \rightarrow Y$  is continuous, then  $(X, \varphi^{-1}(B)) \in \mathcal{K}_a$ . If  $(X, A_1), (X, A_2) \in \mathcal{K}_a$ , then  $(X, A_1 \cap A_2) \in \mathcal{K}_a$ .*

*Proof.* Let  $Z \in \mathcal{K}$  and let  $\psi : Z \rightarrow Y$  be a continuous map with  $\psi(Z) = B$ . Let  $Q$  denote the pullback of the diagram  $X \xrightarrow{\varphi} Y \xleftarrow{\psi} Z$ , i.e.  $Q = \{(x, z) \in X \times Z \mid \varphi(x) = \psi(z)\}$ . Then the projection onto the first coordinate maps  $Q$  onto  $\varphi^{-1}(B)$ . Since  $Q$  is closed in  $X \times Z$ , it is contained in  $\mathcal{K}$ . Hence  $\varphi^{-1}(B)$  is  $\mathcal{K}$ -analytic.

For the second claim, consider the diagonal embedding  $X \rightarrow X \times X$ . The preimage of  $A_1 \times A_2$  is  $A_1 \cap A_2$ . Since  $A_1 \times A_2$  is  $\mathcal{K}$ -analytic, the same is true by the first claim for its preimage  $A_1 \cap A_2$ .  $\square$

For the following lemma recall that  $\mathcal{P}$  is the class of Polish spaces,  $\mathcal{L}^\sigma$  is the class of locally compact and  $\sigma$ -compact spaces, and  $\mathcal{C}$  is the class of compact spaces,

**Lemma 2.6.** *For these classes  $\mathcal{K}$ , the classes  $\mathcal{K}_a$  are as follows:*

- (1)  $\mathcal{P}_a$  consists of all pairs  $(X, A)$ , where  $X$  is Polish and  $A \subseteq X$  is analytic.
- (2)  $\mathcal{L}_a^\sigma$  consists of all pairs  $(X, A)$ , where  $X$  is locally compact  $\sigma$ -compact and  $A \subseteq X$  is  $\sigma$ -compact.
- (3) For the class  $\mathcal{C}$  of compact spaces,  $\mathcal{C}_a$  consists of all pairs  $(X, A)$ , where  $X$  is compact and  $A \subseteq X$  is closed.

*Proof.* Claim (1) is true by definition. For claim (2) we note first of all that the continuous image  $A$  of a  $\sigma$ -compact space  $Z$  is again  $\sigma$ -compact. Conversely, if  $A = \bigcup_{i \geq 0} C_i$  for a countable family of compact sets  $C_i \subseteq X$ , then the topological coproduct (the disjoint union)  $Z = \coprod_{i \geq 0} C_i$  is locally compact  $\sigma$ -compact and maps continuously onto  $A$ . Claim (3) is clear.  $\square$

Recall that a space is called *Lindelöf* if every open covering has a countable subcovering. Both second countable spaces and  $\sigma$ -compact spaces are Lindelöf. In particular, every Polish space and every locally compact  $\sigma$ -compact space is Lindelöf.

**Definition 2.7.** We call a class  $\mathcal{K}$  of spaces which is closed under finite products and under passage to closed subsets *almost Polish* if it has the following three properties.

- (1) Every member of  $\mathcal{K}$  is Lindelöf.
- (2) No member of  $\mathcal{K}$  is meager in itself.
- (3) Every  $\mathcal{K}$ -analytic set is almost open.

**Proposition 2.8.** *The class  $\mathcal{P}$  of Polish spaces, the class  $\mathcal{L}^\sigma$  of locally compact  $\sigma$ -compact spaces, and the class  $\mathcal{C}$  of compact spaces are almost Polish.*

*Proof.* We noted already that every second countable space and every  $\sigma$ -compact space is Lindelöf. Also, every complete metric space and every locally compact space is a Baire



space and hence not meager in itself [6, XI.10.1 and XIV.4.1]. The fact that every analytic set in a Polish space is almost open is proved in [29, p. 482]. Hence  $\mathcal{P}$  is almost Polish. On the other hand, every  $\sigma$ -compact set is clearly a Borel set, and Borel sets are almost open. Therefore  $\mathcal{L}^\sigma$  is almost Polish. For the class  $\mathcal{C}$  we note that in particular every closed set is almost open, hence  $\mathcal{C}$  is almost Polish.  $\square$

For groups in an almost Polish class we have a continuity and open mapping theorem as follows.

**Theorem 2.9.** *Let  $\mathcal{K}$  be a class of almost Polish spaces. Suppose that  $K, G$  are topological groups which belong to  $\mathcal{K}$ , and that*

$$K \xrightarrow{\varphi} G$$

*is an abstract group homomorphism.*

*Assume that for every identity neighborhood  $U \subseteq G$  there exists an identity neighborhood  $V \subseteq U$  such that  $\varphi^{-1}(V)$  is almost open (which is the case if  $\varphi^{-1}(V)$  is  $\mathcal{K}$ -analytic). Then  $\varphi$  is continuous.*

*If the homomorphism  $\varphi$  is continuous and surjective, then  $\varphi$  is open.*

*Proof.* Our proof is based on Pettis' Theorem, which says the following. If  $E$  is a subset of a topological group, and if  $E$  is almost open and not meager, then  $EE^{-1}$  is an identity neighborhood [35, Theorem 1].

Now we prove the first claim. Given an identity neighborhood  $U \subseteq G$ , we choose a smaller identity neighborhood  $V \subseteq U$  such that  $VV^{-1} \subseteq U$ , and such that  $E = \varphi^{-1}(V)$  is almost open. Since  $G$  is Lindelöf, the closed set  $\overline{\varphi(K)}$  is also Lindelöf. Hence there exists a countable set of elements  $g_i \in K$ , for  $i \in \mathbb{N}$ , such that  $\varphi(K) \subseteq \bigcup_{i \geq 0} \varphi(g_i)V$ . Therefore  $K = \bigcup_{i \geq 0} g_i E$ . Since  $K$  is not meager in itself,  $E$  cannot be meager. By Pettis' Theorem,  $EE^{-1}$  is an identity neighborhood in  $K$ , with  $\varphi(EE^{-1}) \subseteq U$ . It follows that  $\varphi$  is continuous at the identity, and hence continuous everywhere [3, III. Proposition 23].

For the second claim let  $W \subseteq K$  be open, let  $g \in W$  and let  $C \subseteq K$  be a closed identity neighborhood with  $gCC^{-1} \subseteq W$ . Such a neighborhood  $C$  exists since every topological group is regular [14, II.4.8]. Then  $\varphi(C) = D$  is  $\mathcal{K}$ -analytic. Since  $K$  is Lindelöf, there exists a countable set of elements  $a_i \in K$ , for  $i \in \mathbb{N}$ , such that  $K = \bigcup_{i \geq 0} a_i C$ . It follows that  $G = \bigcup_{i \geq 0} \varphi(a_i)D$ . Since  $G$  is not meager in itself,  $D$  cannot be meager. By Pettis' Theorem,  $DD^{-1}$  is an identity neighborhood, and hence  $\varphi(g)DD^{-1}$  is a neighborhood of  $\varphi(g)$  which is contained in  $\varphi(W)$ . Thus  $\varphi(W)$  is open.  $\square$

In order to apply Theorem 2.9, we need conditions which ensure that we can construct many  $\mathcal{K}$ -analytic sets. The following lemma supplies such conditions.

**Lemma 2.10.** *Let  $\mathcal{K}$  be a class of spaces which is closed under finite products and the passage to closed subsets and let  $K$  be a topological group in  $\mathcal{K}$ , with a normal (but not necessarily closed)  $\mathcal{K}$ -analytic subgroup  $N$  of  $K$ . Put  $H = K/N$  (as an abstract group)*

and let  $\pi : K \longrightarrow H$  denote the natural quotient homomorphism. Let  $A, B \subseteq H$  be subsets, and assume that  $\pi^{-1}(A)$  and  $\pi^{-1}(B)$  are  $\mathcal{K}$ -analytic. Then the following sets are also  $\mathcal{K}$ -analytic:

- (1) the set  $\pi^{-1}(AB)$ ,
- (2) the set  $\pi^{-1}(\{[a, b] \mid a \in A, b \in B\})$ ,
- (3) all sets  $\pi^{-1}(\{h \in H \mid [h, y] \in A\})$  for  $y \in H$ , and
- (4) all sets  $\pi^{-1}(\text{Cen}_H(X))$  for every finite set  $X \subseteq H$ .

*Proof.* Let  $s : H \longrightarrow K$  be a cross section for  $\pi$ , i.e.  $\pi \circ s = \text{id}_H$ . For every subset  $Y \subseteq H$  we have  $\pi^{-1}(Y) = s(Y)N$ , and  $s(xy)N = s(x)s(y)N$  holds for all  $x, y \in H$ . We note that for all  $\mathcal{K}$ -analytic subsets  $P, Q \subseteq K$ , the product  $PQ \subseteq K$  is again  $\mathcal{K}$ -analytic.

Claim (1) follows from the identity

$$\pi^{-1}(AB) = s(AB)N = s(A)s(B)N = s(A)Ns(B)N = \pi^{-1}(A)\pi^{-1}(B).$$

The rightmost term is  $\mathcal{K}$ -analytic, since products of  $\mathcal{K}$ -analytic sets are again  $\mathcal{K}$ -analytic.

For claim (2) we note that  $\pi^{-1}([a, b]) = s([a, b])N = [s(a), s(b)]N$ , whence

$$\pi^{-1}(\{[a, b] \mid a \in A, b \in B\}) = \{[x, y]n \mid \pi(x) \in A, \pi(y) \in B, n \in N\}.$$

This set is  $\mathcal{K}$ -analytic since products and inverses of  $\mathcal{K}$ -analytic sets in the group  $K$  are again  $\mathcal{K}$ -analytic.

Claim (3) follows from the identity

$$\pi^{-1}(\{h \in H \mid [h, y] \in A\}) = \{g \in K \mid [g, s(y)] \in AN\}.$$

The right-hand side is  $\mathcal{K}$ -analytic by Lemma 2.5.

For (4) we note that  $\pi^{-1}(\text{Cen}_H(x)) = \{g \in K \mid [g, s(x)] \in N\}$  by (3), and this set is  $\mathcal{K}$ -analytic. Since a finite intersection of  $\mathcal{K}$ -analytic sets is again  $\mathcal{K}$ -analytic by Lemma 2.5, claim (4) follows.  $\square$

### 3. BUILDING UP GOOD NEIGHBORHOODS IN LIE GROUPS

Our basic strategy for proving rigidity of Lie groups in almost Polish classes is to construct a neighborhood basis of the identity in the Lie groups which consists of sets that arise in a purely group-theoretic way.

Guided by ideas which go back to van der Waerden, we proceed in two steps. In the first step we construct one specific compact identity neighborhood. In the second step we show that we may ‘shrink’ this given neighborhood in a purely group-theoretic way to an arbitrarily small compact identity neighborhood. This gives us the desired neighborhood basis of the identity. The whole method depends on the existence of nontrivial commutators, both in the Lie algebra and in the Lie group.

**3.1. Lie groups.** Our convention is that a *Lie group*  $G$  is a locally compact group which is a smooth real manifold, such that multiplication and inversion are smooth maps. A priori, no countability assumptions will be imposed on Lie groups. We remark that a connected Lie group is automatically second countable and  $\sigma$ -compact. In particular, a Lie group  $G$  is  $\sigma$ -compact if and only if  $G^\circ$  has countable index in  $G$ , or equivalently, if  $G$  is second countable. The identity component of a Lie group is open, and every closed subgroup of a Lie group is again a Lie group. This fact will be used without further mention. The Lie algebra of a Lie group  $G$  is denoted by  $\text{Lie}(G)$ .

An *analytic Lie subgroup* of a Lie group  $G$  is a subgroup of the form  $H = \langle \exp(\mathfrak{h}) \rangle$ , where  $\mathfrak{h} \subseteq \text{Lie}(G)$  is a Lie subalgebra.<sup>2</sup> Such an analytic Lie subgroup is path connected, but not necessarily closed. However, there exists always a connected Lie group  $L$  and an injective Lie group homomorphism  $\iota : L \rightarrow G$  with  $\iota(L) = H$ . The Lie group  $L$  is unique up to canonical isomorphism [43, 3.19]. Note that the corestriction  $L \rightarrow H$  of  $\iota$  is continuous and bijective, but may fail to be open. The unique Lie subalgebra  $\mathfrak{h}$  of  $\text{Lie}(G)$  is called *the Lie algebra of  $H$* .

**3.2. Spacious sets.** Let  $G$  be a topological group and let  $C \subseteq G$  be a subset. We call  $C$  *spacious* if some product of finitely many left translates of  $CC^{-1}$  has nonempty interior. In the latter case there exist thus elements  $g_1, \dots, g_r \in G$ , for some  $r \geq 1$ , such that

$$D = g_1CC^{-1}g_2CC^{-1} \cdots g_rCC^{-1}$$

is an identity neighborhood. Note that the property of being spacious is inherited if we pass to a larger subset.

We prove below that a nonconstant path in a simple Lie group is always spacious. This is basically a Baire category argument, combined with Yamabe's Theorem. We formalize this as follows.

**3.3. The Baire–Yamabe Process.** Let  $G$  be a Lie group. We denote by  $\text{Fin}(G)$  the poset of all finite subsets of  $G$  containing the identity, ordered by inclusion. By  $\text{PCS}(G)$  we denote the poset of all path connected subgroups of  $G$ , again ordered by inclusion.

We define a poset map as follows. Let  $C \subseteq G$  be a fixed subset and let  $P$  denote the path component of the identity in  $CC^{-1}$ . Then  $P$  is symmetric, path connected, and  $1 \in P$ . If  $C$  contains a nonconstant path, then  $P$  also contains a nonconstant path. For any  $F \in \text{Fin}(G)$  we put

$$(*) \quad X_C(F) = \bigcup \{gPg^{-1} \mid g \in F\} \text{ and } Y_C(F) = \langle X_C(F) \rangle.$$

We note that the set  $X_C(F)$  is path connected, symmetric, and contains the identity. Therefore the group

$$Y_C(F) = X_C(F) \cup X_C(F)X_C(F) \cup X_C(F)X_C(F)X_C(F) \cup \cdots$$

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<sup>2</sup> Such a subgroup is also called an *integral subgroup*, a *virtual subgroup* or an *analytic subgroup*—the terminology varies widely.

is also path connected. The assignment

$$Y_C : \text{Fin}(G) \longrightarrow \text{PCS}(G)$$

is a poset map (i.e preserves the partial order  $\subseteq$ ). We call the function  $Y_C$  the *Baire–Yamabe Process*.

Now we recall Yamabe’s Theorem, see [11] or [16, Theorem 9.6.1].

**Theorem 3.4** (Yamabe). *Let  $H$  be a path connected subgroup of a Lie group  $G$ . Then  $H$  is an analytic Lie subgroup. In particular,  $H$  determines a unique subalgebra  $\text{Lie}(H) \subseteq \text{Lie}(G)$ .*

Recall that a partially ordered set or *poset* in short, is a set with a reflexive and transitive relation. A *chain* in a poset is a totally ordered subset. A poset is *inductive* if every chain has an upper bound. In an inductive poset, by Zorn’s Lemma, for any element  $x$  there is a maximal element  $m$  such that  $x \leq m$ . A poset in which every chain is finite is trivially inductive. The set of vector subspaces of a finite dimensional vector space is such an inductive poset under inclusion. Yamabe’s Theorem yields a poset bijection between the poset of path connected subgroups  $\text{PCS}(G)$  and the poset  $\text{LSA}(\text{Lie}(G))$  of all Lie subalgebras of  $\text{Lie}(G)$ , ordered by inclusion. So the poset  $\text{PCS}(G)$  of all path connected subgroups of a Lie group is inductive and the same holds for the image  $\text{im}(Y_C)$  of the Baire–Yamabe Process. Note also that  $P \subseteq Y_C(\{1\}) \subseteq Y_C(F)$  holds for all  $F \in \text{Fin}(G)$ .

In the next proposition we consider an arbitrary Lie group  $G$  and specify the subset  $C \subseteq G$  to be the image of a continuous path  $f : [0, 1] \longrightarrow G$  in  $G$ . We recall that then  $P = CC^{-1}$ . With theses assumptions we have the following result.

**Proposition 3.5.** *Let  $Y_C(F)$  be a maximal member in the poset  $\text{im}(Y_C) \subseteq \text{PCS}(G)$  of path connected subgroups of  $G$  obtained by the Baire–Yamabe Process from  $C = f([0, 1])$ . Then  $Y_C(F)$  is a normal subgroup containing  $P$ , and, accordingly, its Lie algebra  $\text{Lie}(Y_C(F))$  is a  $G$ -invariant ideal in  $\text{Lie}(G)$ .*

*Proof.* For an arbitrary element  $g \in G$  we have  $Y_C(F) \subseteq Y_C(F \cup gF)$  and by the maximality of  $Y_C(F)$  in  $\text{im}(Y_C)$ , equality holds. Hence by  $(*)$  we have

$$\begin{aligned} gY_C(F)g^{-1} &= g \left( \bigcup \{hPh^{-1} \mid h \in F\} \right) g^{-1} \\ &\subseteq \bigcup \{hPh^{-1} \mid h \in F \cup gF\} = X_C(F \cup gF) \\ &\subseteq Y_C(F \cup gF) = Y_C(F). \end{aligned}$$

Thus  $gY_C(F)g^{-1} \subseteq Y_C(F)$  and therefore  $Y_C(F)$  is a normal subgroup. It follows that  $\text{Lie}(Y_C(F))$  is an ideal in  $\text{Lie}(G)$ , see [21, Proposition 5.54(i)] or [16, Corollary 11.1.3]. Since the analytic Lie subgroup  $Y_C(F)$  is normal in  $G$ , its Lie algebra is invariant under the adjoint action of  $G$ .  $\square$

The following proposition clarifies the role of spacious paths for the Baire–Yamabe Process.

**Proposition 3.6.** *Let  $G$  be a Lie group and  $f : [0, 1] \rightarrow G$  a path. Put  $C = f([0, 1])$ . Then the following two conditions are equivalent:*

- (1)  $C$  is spacious in  $G$ .
- (2) There is a finite set  $F \in \text{Fin}(G)$  such that  $Y_C(F) = G^\circ$ .

*Proof.* We denote the  $k$ -fold product of a set  $A \subseteq G$  by  $A^k$ . Set  $P = CC^{-1}$  and assume (1). Then there exist elements  $g_1, \dots, g_r \in G$  such that

$$D = g_1 P g_2 \cdots g_r P$$

is an identity neighborhood. Define elements  $h_1, \dots, h_r$  recursively so that  $h_1 = g_1$  and  $h_j = h_{j-1} g_j$ , for  $j = 2, \dots, r$ . Put  $F = \{1, h_1, \dots, h_r\}$ . Then  $F \in \text{Fin}(G)$  and by  $P = P^{-1}$  and (\*) we have

$$\begin{aligned} D &\subseteq DD^{-1} = (h_1 P h_1^{-1})(h_2 P h_2^{-1}) \cdots (h_r P)(P h_r^{-1}) \cdots (h_1 P h_1^{-1}) \\ &= (h_1 P h_1^{-1}) \cdots (h_r P h_r^{-1})(h_r P h_r^{-1}) \cdots (h_1 P h_1^{-1}) \subseteq X_C(F)^{2r} \subseteq Y_C(F). \end{aligned}$$

The subgroup  $Y_C(F)$  is path connected on the one hand, whence  $Y_C(F) \subseteq G^\circ$ , and it is open since it contains the identity neighborhood  $D$ . Therefore it is an open and closed subgroup, whence  $G^\circ \subseteq Y_C(F)$ . This proves (2).

Now assume (2). Then there exists  $F \in \text{Fin}(G)$  such that  $G^\circ = Y_C(F) = \bigcup_{n=1}^{\infty} X_C(F)^n$ . Each of the sets  $X_C(F)^n$  is compact. By Baire's Category Theorem [6, XI.10.3] there is an  $n$  such that  $X_C(F)^n$  has nonempty interior. Now

$$\begin{aligned} X_C(F)^n &= \left( \bigcup_{h_1 \in F} h_1 P h_1^{-1} \right) \cdots \left( \bigcup_{h_n \in F} h_n P h_n^{-1} \right) \\ &= \bigcup \{ h_1 P h_1^{-1} h_2 P h_2^{-1} \cdots h_n P h_n^{-1} \mid h_1, \dots, h_n \in F \}. \end{aligned}$$

The Baire Category Theorem, applied once more to this finite union of compact sets, shows that there exist elements  $h_1, h_2, \dots, h_n \in F$  such that the set

$$E = h_1 P h_1^{-1} h_2 P h_2^{-1} \cdots h_n P h_n^{-1}$$

has a nonempty interior. Setting  $g_1 = h_1$ ,  $g_2 = h_1^{-1} h_2$ ,  $\dots$ ,  $g_n = h_{n-1}^{-1} h_n$  in  $G$ , we see that  $E h_n = g_1 P g_2 \cdots g_n P$  has nonempty interior, whence  $C$  is spacious in  $G$  by definition.  $\square$

**Proposition 3.7.** *Suppose that  $G$  is a Lie group whose Lie algebra is simple, and that*

$$f : [0, 1] \rightarrow G$$

*is a continuous nonconstant path. Then  $C = f([0, 1])$  is spacious in  $G$ .*

*Proof.* Put  $P = CC^{-1}$  and let  $Y_C(F)$  be a maximal element resulting from the Baire–Yamabe process. The Lie algebra  $\text{Lie}(G)$  is simple and  $\{1\} \neq P$ , hence  $Y_C(F) \neq \{1\}$ . By Proposition 3.5, the Lie algebra  $\text{Lie}(Y_C(F))$  is a nonzero ideal in  $\text{Lie}(G)$ . Hence it equals  $\text{Lie}(G)$ , and thus  $Y_C(F) = G^\circ$ . The claim follows now from Proposition 3.6.  $\square$

Now we show how to shrink compact identity neighborhoods in perfect Lie groups in an algebraic way, using commutators. This method is basically due to van der Waerden [42]. The present approach follows closely [21, 5.59], but avoids the BCH multiplication in Banach algebras.

We start with a lemma about Lie algebras.

**Lemma 3.8.** *Let  $Z_1 = [X_1, Y_1], \dots, Z_n = [X_n, Y_n]$  be  $n$  linearly independent commutators in a finite dimensional real Lie algebra  $\mathfrak{g}$ . Then there exists a real number  $r > 0$  such that the  $n$  vectors*

$$\tilde{Z}_i(t) = \exp(\text{ad}(t_i X_i))Y_i - Y_i$$

*are linearly independent, for all  $0 < |t_i| \leq r$ .*

*Proof.* In the  $n$ th exterior power of the Lie algebra  $\mathfrak{g}$  we have  $Z_1 \wedge \dots \wedge Z_n \neq 0$ . From the series expansion  $\exp(\text{ad}(tX)) = \sum_{k=0}^{\infty} \frac{1}{k!}(\text{ad}(tX))^k$  we see that

$$\frac{1}{t}(\exp(\text{ad}(tX))Y - Y) = [X, Y] + tF(t, X, Y),$$

for some continuous function  $F : \mathbb{R} \times \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ . Put  $Z_k(t) = \frac{1}{t}(\exp(\text{ad}(tX_k))Y_k - Y_k)$ , with  $Z_k(0) = Z_k$ . Thus  $Z_k(t)$  depends continuously on  $t$ . By the continuity of the map  $(t_1, \dots, t_n) \mapsto Z_1(t_1) \wedge \dots \wedge Z_n(t_n)$  at  $(0, \dots, 0)$ , there exists a constant  $r > 0$  such that  $Z_1(t_1) \wedge \dots \wedge Z_n(t_n) \neq 0$  for all  $|t_1|, \dots, |t_n| \leq r$ . Hence

$$\tilde{Z}_1(t_1) \wedge \dots \wedge \tilde{Z}_n(t_n) = t_1 \dots t_n Z_1(t_1) \wedge \dots \wedge Z_n(t_n) \neq 0,$$

provided that  $t_1 \dots t_n \neq 0$  and  $|t_1|, \dots, |t_n| \leq r$ .  $\square$

The next fact we need from point-set topology is well-known and follows from the Tube Lemma [6, XI.2.6].

**Lemma 3.9** (Wallace). *Let  $X, Y, Z$  be spaces and let  $\varphi : X \times Y \rightarrow Z$  be a continuous map. Let  $A$  and  $B$  be compact subsets of  $X$  and  $Y$ , respectively. Suppose that  $W$  is an open set of  $Z$  containing  $\varphi(A \times B)$ . Then there exist neighborhoods  $U$  of  $A$  in  $X$  and  $V$  of  $B$  in  $Y$  such that  $\varphi(U \times V) \subseteq W$ .*

We obtain the following variation of van der Waerden’s Theorem [42].

**Theorem 3.10** ([21, Proposition 5.59]). *Let  $G$  be an  $n$ -dimensional Lie group whose Lie algebra is perfect, i.e.  $\text{Lie}(G)$  is spanned by commutators. Then there exist 1-parameter groups  $c_1, \dots, c_n$  in  $G$  such that for all  $t_1, \dots, t_n$  with  $0 < |t_i| \leq 1$  and every identity neighborhood  $U \subseteq G$ , the set*

$$M_{t_1, \dots, t_n} = [c_1(t_1), U] \dots [c_n(t_n), U] \subseteq G$$

is an identity neighborhood. If  $U$  is compact and if  $W \subseteq G$  is any identity neighborhood, then the  $t_i > 0$  can be chosen in such a way that  $M_{t_1, \dots, t_n} \subseteq W$ .

*Proof.* Let  $Z_1 = [X_1, Y_1], \dots, Z_n = [X_n, Y_n]$  be a basis of the Lie algebra  $\text{Lie}(G)$ . By Lemma 3.8 there exists a number  $r > 0$  such that for all  $t_1, \dots, t_n$  with  $0 < |t_i| \leq r$  the  $n$  vectors  $\tilde{Z}_i(t) = \exp(\text{ad}(t_i X_i))Y_i - Y_i$  form a basis of  $\text{Lie}(G)$ . Put  $c_i(t) = \exp(rtX_i)$  and  $y_i(t) = \exp(rtY_i)$ .

Suppose that  $0 < |t_i| \leq 1$ . Differentiating the smooth function  $s \mapsto [c_i(t_i), y_i(s)]$  at the time  $s = 0$ , we obtain the vector  $\tilde{Z}_i(t_i) = \exp(\text{ad}(t_i X_i))Y_i - Y_i$ . From the inverse function theorem we conclude that near  $(0, \dots, 0)$  the map

$$(s_1, \dots, s_n) \mapsto [c_1(t_1), y_1(s_1)] \cdots [c_n(t_n), y_n(s_n)]$$

is a diffeomorphism from  $\mathbb{R}^n$  to  $G$ . Hence  $M = [c_1(t_1), U] \cdots [c_n(t_n), U]$  is an identity neighborhood. If  $U$  is compact, then by Wallace's Lemma 3.9 we may choose the  $t_i$  in such a way that  $M \subseteq W$ .  $\square$

**3.11. Remark.** We note the following consequence of Theorem 3.10. In an  $n$ -dimensional Lie group  $G$  whose Lie algebra is perfect, every element near the identity is a product of at most  $n$  commutators of elements coming from a small identity neighborhood. In particular, the abstract commutator subgroup of  $G$  is open in  $G$  (this follows also from Yamabe's Theorem 3.4).

#### 4. RIGIDITY OF SEMISIMPLE LIE GROUPS

The following result is essential for almost all our rigidity results concerning Lie groups.

**Theorem 4.1.** *Let  $\mathcal{K}$  be an almost Polish class and suppose that  $G$  is a Lie group in  $\mathcal{K}$  whose Lie algebra is perfect. Suppose also that*

$$1 \longrightarrow N \hookrightarrow K \xrightarrow{\varphi} G \longrightarrow 1$$

*is a short exact sequence, for some abstract homomorphism  $\varphi$ , and that  $(K, N) \in \mathcal{K}_a$ . If there exists a compact spacious subset  $C \subseteq G$  whose preimage  $\varphi^{-1}(C)$  is  $\mathcal{K}$ -analytic, then  $\varphi$  is continuous and open.*

*Proof.* By our assumptions on  $C$  we find elements  $g_1, \dots, g_r \in G$  such that the compact set  $D = g_1 C C^{-1} g_2 C C^{-1} \cdots g_r C C^{-1}$  is a compact identity neighborhood. An iterated application of Lemma 2.10(1) shows that its preimage  $\varphi^{-1}(D)$  is  $\mathcal{K}$ -analytic. Let  $W \subseteq G$  be an arbitrary identity neighborhood. By Theorem 3.10 we can find elements  $a_1, \dots, a_n \in G$  such that  $M = [a_1, D] \cdots [a_n, D] \subseteq W$  is an identity neighborhood. Now we apply Lemma 2.10(2) and (1) and conclude that its preimage  $\varphi^{-1}(M)$  is  $\mathcal{K}$ -analytic. By Theorem 2.9, the homomorphism  $\varphi$  is continuous and open.  $\square$

The following immediate consequence will be generalized below. Under the additional assumption that  $\varphi$  is an isomorphism,  $G$  is connected and that  $\mathcal{K}$  is  $\mathcal{L}^\sigma$  or  $\mathcal{P}$ , such a result is proved in [9, 24, 25], respectively.

**Corollary 4.2.** *Let  $G$  be a compact Lie group whose Lie algebra is semisimple. Then  $G$  is rigid within every almost Polish class  $\mathcal{K}$  that contains  $G$ .*

*Proof.* We apply Theorem 4.1 to the spacious set  $C = G$ .  $\square$

The following well-known fact about Lie groups will be used several times.

**Lemma 4.3.** *Let  $G$  be a Lie group. If  $G/G^\circ$  is finitely generated, then  $G$  has a finitely generated dense subgroup.*

*Proof.* It suffices to consider the case where  $G$  is connected, and we proceed by induction on the dimension  $n$  of the Lie group  $G$ . The case  $n = 0$  is trivial. In general, let  $H \subseteq G$  be a maximal closed connected proper subgroup in  $G$ . Such subgroups exist, since  $\dim(G)$  is finite, and  $\dim(H) < \dim(G)$ . By the induction hypothesis, there exists a finite set  $X \subseteq H$  which generates a dense subgroup of  $H$ . Let  $c : \mathbb{R} \rightarrow G$  be a 1-parameter group whose image is not contained in  $H$ . Then  $c(\mathbb{R}) \cup H$  generates a connected subgroup  $L$  of  $G$ . Hence the closure  $\overline{L}$  of  $L$  is a closed connected subgroup. Thus  $\overline{L} = G$  by the maximality of  $H$ . The real numbers  $1, \sqrt{2} \in \mathbb{R}$  generate additively a dense subgroup in  $\mathbb{R}$ . Hence  $\{c(1), c(\sqrt{2})\} \cup X$  generates a dense subgroup in  $G$ .  $\square$

The following observation will be used below. If a Lie group  $G$  acts continuously on a space  $X$ , and if the stabilizer  $G_x$  of the point  $x \in X$  is not open (i.e. if  $G^\circ \not\subseteq G_x$ ), then the orbit  $G(x) \subseteq X$  contains a nonconstant path. For we may choose then a 1-parameter group  $c : \mathbb{R} \rightarrow G$  which is not contained in  $G_x$ . Then  $c([0, 1])$  is not contained in  $G_x$ , and thus  $t \mapsto c(t)(x)$  is a nonconstant path in the orbit  $G(x)$ .

We recall from the introduction that a simple real Lie algebra  $\mathfrak{g}$  is called *absolutely simple* if its complexification  $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$  is a simple complex Lie algebra. We extract the following technical result from [28].

**Proposition 4.4.** *Let  $G$  be a connected Lie group whose Lie algebra is absolutely simple. Assume also that the center of  $G$  is finite. Then there exists an element  $h \in G$  and a finite subset  $X \subseteq G$  such that the set  $C = \{ghg^{-1} \mid g \in \text{Cen}_G(X)\}$  is compact and contains a nonconstant path. In particular,  $C$  is spacious in  $G$ .*

*Proof.* If  $G$  is compact, we put  $X = \{1\}$  and we choose  $h \in G - \text{Cen}(G)$ . Then the conjugacy class  $C = \{ghg^{-1} \mid g \in G\}$  is compact and, since  $h$  is not central, contains a nonconstant path by the observation recorded above. By Proposition 3.7, the set  $C$  is spacious in  $G$ .

Suppose now that  $G$  is not compact. Then  $G$  has positive real rank and, in particular, there exist nontrivial parabolic subgroups in  $G$ . We refer to [28, pp. 2627–2628] for the



following facts. In *loc.cit.* it is assumed that  $G$  is centerless, but the reasoning remains valid in the presence of a finite center, as we explain now.

Let  $P \subseteq G$  be a next-to-minimal parabolic, i.e. up to conjugation, there is exactly one parabolic subgroup of  $G$  properly contained in  $P$ . Let  $H \subseteq P$  be a reductive Levi subgroup, so that  $P = HU$ , where  $U$  is the unipotent radical of  $P$ . Then  $H$  is a reductive group of real rank 1 (because  $P$  was next-to-minimal). It is shown in [44, p. 73]<sup>3</sup> that the group  $H$  can be written as the  $G$ -centralizer of a closed connected abelian subgroup  $S \subseteq G$ . Since  $\text{Lie}(G)$  is absolutely simple, the parabolic  $P$  can be chosen in such a way that  $\mathfrak{sl}_2\mathbb{C}$  is not a direct factor in  $\text{Lie}(H)$ , see [28, Lemma 10]. It is shown in *loc.cit.* p. 2627 that then there exists a closed connected abelian subgroup  $T \subseteq G$  and an element  $h \in G$  such that the conjugates of  $h$  under the group  $L = \text{Cen}_H(T)$  form a compact set containing a nonconstant path. Now

$$L = H \cap \text{Cen}_G(T) = \text{Cen}_G(S) \cap \text{Cen}_G(T) = \text{Cen}_G(\overline{\langle S \cup T \rangle}).$$

By Lemma 4.3 we may choose a finite set  $X \subseteq \overline{\langle S \cup T \rangle}$  which generates a dense subgroup of  $\overline{\langle S \cup T \rangle}$ . Thus  $L = \text{Cen}_G(X)$ , and  $C = \{ghg^{-1} \mid g \in L\}$  is compact and contains a nonconstant path, as required. By Proposition 3.7, the set  $C$  is spacious in  $G$ .  $\square$

If we combine Proposition 4.4 with Theorem 4.1, we obtain immediately a rigidity result for Lie groups satisfying the assumptions of Proposition 4.4. However, we can do better. First we extend Proposition 4.4 to the semisimple case.

**Proposition 4.5.** *Let  $G$  be a Lie group. Suppose that  $\text{Cen}(G^\circ)$  is finite and that the Lie algebra  $\text{Lie}(G)$  is a direct sum of absolutely simple ideals. Then there exists an element  $h$  and a finite subset  $X$  in  $G^\circ$  such that  $C = \{ghg^{-1} \mid g \in \text{Cen}_G(X)\}$  is compact, spacious, and contains a nonconstant path.*

*Proof.* We first consider the case where  $G$  is connected. Then  $G$  is a connected semisimple Lie group with finite center. We proceed by induction on the number  $r$  of simple ideals in the Lie algebra of  $G$ . The case  $r = 1$  is taken care of by Proposition 4.4. For  $r \geq 2$  we decompose the Lie algebra of  $G$  into a simple ideal and a complementary semisimple ideal. Accordingly,  $G$  is a central product  $G = G_1G_2$  of two closed connected commuting subgroups  $G_1, G_2$ , with  $\text{Lie}(G_1)$  absolutely simple and where  $G_2$  has a semisimple Lie algebra which is a sum of  $r - 1$  absolutely simple ideals. By the induction hypothesis we find elements  $h_i \in G_i$  and finite subsets  $X_i \subseteq G_i$  such that the sets  $C_i = \{gh_i g^{-1} \mid g \in \text{Cen}_{G_i}(X_i)\}$  contain nonconstant paths and are compact and spacious in  $G_i$ , for  $i = 1, 2$ . Since  $[G_1, G_2] = 1$  and  $G = G_1G_2$ , we have  $C_i = \{gh_i g^{-1} \mid g \in \text{Cen}_G(X_i)\}$ . Put  $h = h_1h_2$

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<sup>3</sup> Where  $H = L_\Theta$  in the notation of *loc.cit.*

and  $X = X_1 \cup X_2$  and  $C = \{ghg^{-1} \mid g \in \text{Cen}_G(X)\}$ . Then

$$\begin{aligned} C &= \{(g_1g_2)(h_1h_2)(g_1g_2)^{-1} \mid g_i \in G_i \text{ for } i = 1, 2 \text{ and } g_1g_2 \in \text{Cen}_G(X)\} \\ &= \{(g_1g_2)(h_1h_2)(g_1g_2)^{-1} \mid g_i \in \text{Cen}_{G_i}(X_i) \text{ for } i = 1, 2\} \\ &= \{(g_1h_1g_1^{-1})(g_2h_2g_2^{-1}) \mid g_i \in \text{Cen}_{G_i}(X_i) \text{ for } i = 1, 2\} \\ &= C_1C_2 \end{aligned}$$

is compact and contains a nonconstant path. There exist elements  $g_{i,1}, \dots, g_{i,s} \in G_i$  such that  $D_i = g_{i,1}C_iC_i^{-1}g_{i,2}C_iC_i^{-1} \cdots g_{i,s}C_iC_i^{-1}$  is a compact identity neighborhood in  $G_i$ , for  $i = 1, 2$ . We can safely assume that the number  $s$  of elements is in both cases the same, because the product of an open set and an arbitrary set in a topological group is always open. Consider the natural surjective homomorphism  $G_1 \times G_2 \rightarrow G$ . This map is open and maps  $D_1 \times D_2$  onto  $D_1D_2$ , hence  $D_1D_2$  is an identity neighborhood in  $G$ . Put  $g_j = g_{1,j}g_{2,j}$ , for  $j = 1, \dots, s$ . Then

$$D_1D_2 = g_1CC^{-1}g_2CC^{-1} \cdots g_sCC^{-1}.$$

Thus  $C$  is spacious in  $G$ .

It remains to consider the case where  $G$  is not connected. Then  $G^\circ$  is a connected semisimple Lie group with finite center, as above. We choose  $h$  and  $X$  in  $G^\circ$  as before and we put

$$C_0 = \{ghg^{-1} \mid g \in \text{Cen}_{G^\circ}(X)\} \subseteq \{ghg^{-1} \mid g \in \text{Cen}_G(X)\} = C.$$

The set  $C_0$  is thus compact, spacious, and contains a nonconstant path. It remains to show that  $C$  is compact. Consider the Lie group  $\text{Aut}_{\mathbb{R}}(\text{Lie}(G))$  of all  $\mathbb{R}$ -linear automorphisms of the Lie algebra  $\text{Lie}(G)$ . The group  $G$  acts on  $\text{Lie}(G)$  via the adjoint representation

$$G \xrightarrow{\text{Ad}} \text{Aut}_{\mathbb{R}}(\text{Lie}(G)).$$

Put  $H = \text{Ad}(G)$ . We note that  $H$  acts faithfully on  $G^\circ$  by conjugation, because  $G^\circ$  is connected. Under the homomorphism  $\text{Ad}$ , the connected group  $G^\circ$  maps onto the identity component  $\text{Aut}_{\mathbb{R}}(\text{Lie}(G))^\circ$ , because  $\text{Lie}(G)$  is semisimple. Hence  $\text{Ad}(G^\circ) = H^\circ$ . The quotient  $\text{Aut}_{\mathbb{R}}(\text{Lie}(G))/\text{Aut}_{\mathbb{R}}(\text{Lie}(G))^\circ$  is finite, see [32, Corollary 2], [16, Proposition 13.1.5], or [12] for a stronger structural result. Hence  $[H : H^\circ]$  is also finite. Let  $H_X$  denote the point-wise stabilizer of  $X$  under the  $H$ -action on  $G^\circ$ . Then  $(H^\circ)_X = H_X \cap H^\circ$  has finite index in  $H_X$ . Hence  $[\text{Cen}_G(X) : \text{Cen}_{G^\circ}(X)]$  is finite as well, and we may put

$$\text{Cen}_G(X) = a_1 \text{Cen}_{G^\circ}(X) \cup \cdots \cup a_k \text{Cen}_{G^\circ}(X),$$

for elements  $a_1, \dots, a_k \in G$ . Thus

$$C = a_1C_0a_1^{-1} \cup \cdots \cup a_kC_0a_k^{-1}$$

is compact. □

The following result is Theorem A in the Introduction.

**Theorem 4.6.** *Let  $G$  be a Lie group. Suppose that  $\text{Cen}(G^\circ)$  is finite and that the Lie algebra  $\text{Lie}(G)$  is a direct sum of absolutely simple ideals. Let  $\mathcal{K}$  be an almost Polish class containing  $G$ . Then  $G$  is rigid within  $\mathcal{K}$ .*

*Proof.* Let  $h$  and  $X$  be as in Proposition 4.5 and put  $C = \{ghg^{-1} \mid g \in \text{Cen}_G(X)\}$ . Then  $C$  is compact and spacious by *loc.cit.* Suppose that

$$1 \longrightarrow N \hookrightarrow K \xrightarrow{\varphi} G \longrightarrow 1$$

is a short exact sequence, for some abstract homomorphism  $\varphi$ , and that  $(K, N) \in \mathcal{K}_a$ . The preimage of  $C$  is  $\mathcal{K}$ -analytic by Lemma 2.10(4). Hence  $G$  is rigid by Theorem 4.1.  $\square$

## 5. THE CASE OF SEMIDIRECT PRODUCTS OF LIE GROUPS

In this section, all Lie group representations will be assumed to be finite dimensional and continuous, unless stated otherwise.

**5.1. Construction.** Given a real representation  $\rho : H \longrightarrow \text{GL}(V)$  of a Lie group  $H$  we may form the semidirect product

$$G = V \rtimes_{\rho} H.$$

The underlying manifold is  $V \times H$ , and the multiplication is given by

$$(**) \quad (u, a)(v, b) = (u + av, ab),$$

where we write  $\rho(a)(v) = av$  for short. The neutral element is  $(0, 1)$  and the inverse of  $(u, a)$  is

$$(u, a)^{-1} = (-a^{-1}u, a^{-1}).$$

Then clearly  $G$  is a Lie group. We may identify  $H$  and  $V$  with closed subgroups of  $G$ , and  $V$  is normal in  $G$ .

As a vector space, the Lie algebra of  $G$ , which is the tangent space of the manifold  $V \times H$  at the point  $(0, 1)$ , is given by  $\text{Lie}(G) = V \oplus \text{Lie}(H)$ . From this decomposition and the representations  $\text{Ad}$  and  $\rho$  of  $H$  on  $\text{Lie}(H)$  and  $V$ , the Lie bracket can be worked out as

$$[(u, X), (v, Y)] = (Xv - Yu, [X, Y]),$$

where  $u, v \in \text{Lie}(V) = V$  and  $X, Y \in \text{Lie}(H)$ , and where we put  $Xv = \text{Lie}(\rho)(X)(v)$  for short. We refer to [15, Chapter V.3] for details.

Now we consider Lie subalgebras of  $\text{Lie}(G)$  containing  $\text{Lie}(H)$ . As an  $H^\circ$ -module under the adjoint action, the vector space  $\text{Lie}(G)$  decomposes as a direct sum of  $H^\circ$ -modules as

$$\text{Lie}(G) = V \oplus \text{Lie}(H).$$

Suppose that  $\mathfrak{m} \subseteq \text{Lie}(G)$  is a Lie subalgebra containing  $\text{Lie}(H)$ . Since  $\mathfrak{m}$  is then an  $H^\circ$ -module, we conclude that

$$\mathfrak{m} = (\mathfrak{m} \cap V) \oplus \text{Lie}(H)$$

as an  $H^\circ$ -module. This observation has the following consequences. We use the notation set up so far.

**Lemma 5.2.** *Let  $\rho : H \longrightarrow \mathrm{GL}(V)$  be a real representation of a Lie group  $H$  and put  $G = V \rtimes_\rho H$ .*

- (1) *If the subrepresentation  $H^\circ \longrightarrow \mathrm{GL}(V)$  is nontrivial, then  $\mathrm{Lie}(H)$  is not an ideal in  $\mathrm{Lie}(G)$ .*
- (2) *If the subrepresentation  $H^\circ \longrightarrow \mathrm{GL}(V)$  is irreducible, then  $\mathrm{Lie}(H)$  is a maximal proper subalgebra in  $\mathrm{Lie}(G)$ .*
- (3) *If  $\mathrm{Lie}(H)$  is perfect and if the subrepresentation  $H^\circ \longrightarrow \mathrm{GL}(V)$  is nontrivial and irreducible, then  $\mathrm{Lie}(G)$  is perfect.*

*Proof.* If the connected Lie group  $H^\circ$  acts nontrivially on  $V$  via  $\rho$ , then its Lie algebra also acts nontrivially on  $V$  via  $\mathrm{Lie}(\rho)$ . Hence there exists  $X \in \mathrm{Lie}(H)$  and  $v \in V$  with  $Xv \neq 0$ . Then  $[(0, X), (v, 0)] = (Xv, 0) \notin \mathrm{Lie}(H)$ , hence  $\mathrm{Lie}(H)$  is not an ideal. This proves (1).

For (2), suppose that  $\mathfrak{m}$  is a subalgebra of  $\mathrm{Lie}(G)$  containing  $\mathrm{Lie}(H)$  properly. Then  $V \cap \mathfrak{m} \neq 0$  as we noted above. Since  $V$  is by assumption an irreducible  $H^\circ$ -module,  $V \cap \mathfrak{m} = V$  and hence  $\mathfrak{m} = \mathrm{Lie}(G)$ .

If  $\mathrm{Lie}(H)$  is perfect, then  $\mathrm{Lie}(H) = [\mathrm{Lie}(H), \mathrm{Lie}(H)] \subseteq [\mathrm{Lie}(G), \mathrm{Lie}(G)]$ . By (1), the Lie algebra  $\mathrm{Lie}(H)$  is not an ideal in  $\mathrm{Lie}(G)$ , whence  $\mathrm{Lie}(H) \neq [\mathrm{Lie}(G), \mathrm{Lie}(G)]$ . By (2), this implies that  $[\mathrm{Lie}(G), \mathrm{Lie}(G)] = \mathrm{Lie}(G)$ .  $\square$

**Lemma 5.3.** *Suppose that  $\rho : H \longrightarrow \mathrm{GL}(V)$  is a real representation of a Lie group  $H$ , that  $\mathrm{Lie}(H)$  is perfect and that the subrepresentation  $H^\circ \longrightarrow \mathrm{GL}(V)$  is nontrivial and irreducible. Let  $f : [0, 1] \longrightarrow H$  be a path and put  $C = f([0, 1])$ . If  $C$  is spacious in  $H$ , then  $C$  is also spacious in  $G$ .*

*Proof.* We consider first the Baire–Yamabe Process  $Y_C$  in  $H$ . By Lemma 3.6 there is a finite set  $E \in \mathrm{Fin}(H)$  with

$$Y_C(E) = H^\circ.$$

Note that this equation remains valid if we view  $E$  as a subset of the larger group  $G = V \rtimes_\rho H$ .

Now we consider the Baire–Yamabe Process  $Y_C$  in  $G$ . Let  $Y_C(F) \subseteq G$  be a maximal element in  $\mathrm{im}(Y_C)$ . Thus  $Y_C(F) = Y_C(E \cup F) \supseteq Y_C(E)$ . Moreover,  $\mathrm{Lie}(Y_C(F))$  is by Proposition 3.5 an ideal in  $\mathrm{Lie}(G)$ . Therefore  $\mathrm{Lie}(Y_C(F)) = \mathrm{Lie}(G)$  by Lemma 5.2(3), and thus  $Y_C(F) = G^\circ$ . Again by Lemma 3.6, the compact set  $C$  is spacious in  $G$ .  $\square$

For the next theorem we first recall Schur’s Lemma. If  $M \subseteq \mathrm{GL}(V)$  is a subgroup acting irreducibly on the finite dimensional real vector space  $V$ , then the ring  $\mathbb{D} = \mathrm{End}_M(V)$  of all endomorphisms of  $V$  that commute with the  $M$ -action is a finite dimensional division ring over  $\mathbb{R}$ , and hence

$$\mathbb{D} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\},$$

where  $\mathbb{H}$  denotes the division ring of real quaternions. The vector space  $V$  then becomes a right  $\mathbb{D}$ -module, and  $M$  acts as a group of  $\mathbb{D}$ -linear endomorphisms.

Suppose that  $V$  is a finite dimensional right  $\mathbb{D}$ -module, for  $\mathbb{D} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ . Let  $|\cdot|$  denote the standard norm on  $\mathbb{D}$  and let

$$U = \{u \in \mathbb{D} \mid |u| = 1\}$$

denote the compact group of norm 1 elements in  $\mathbb{D}^*$ . Every element  $a \in \mathbb{D}$  has a (unique) presentation in polar coordinates as  $a = ut$ , with  $u \in U$  and  $t = |a|$ . Put

$$\mathrm{SL}_{\mathbb{D}}(V) = [\mathrm{GL}_{\mathbb{D}}(V), \mathrm{GL}_{\mathbb{D}}(V)].$$

Then there is a short exact sequence of linear Lie groups

$$1 \longrightarrow \mathrm{SL}_{\mathbb{D}}(V) \hookrightarrow \mathrm{GL}_{\mathbb{D}}(V) \xrightarrow{\det_{\mathbb{D}}} K_1(\mathbb{D}) \longrightarrow 1.$$

For the commutative fields  $\mathbb{D} = \mathbb{R}$  and  $\mathbb{D} = \mathbb{C}$ , the range of  $\det_{\mathbb{D}}$  is  $K_1(\mathbb{D}) = \mathbb{D}^*$  and  $\det_{\mathbb{D}}$  is the usual determinant. In the quaternionic case  $K_1(\mathbb{H}) = \mathbb{R}_{>0}$  is the multiplicative groups of the positive reals and  $\det_{\mathbb{H}}$  is the Dieudonné determinant, which is given by

$$\det_{\mathbb{H}} = |\det_{\mathbb{C}}|,$$

where  $V$  is viewed as as right  $\mathbb{C}$ -module.

If  $V$  is a finite dimensional right  $\mathbb{D}$ -module, we may view  $\mathbb{D}^*$  as a closed subgroup of  $\mathrm{GL}_{\mathbb{R}}(V)$ , by identifying  $a \in \mathbb{D}^*$  with the linear map  $v \mapsto va$ . Then the product

$$\mathrm{SL}_{\mathbb{D}}(V)\mathbb{D}^* = \mathrm{GL}_{\mathbb{D}}(V)\mathbb{D}^*$$

is a subgroup of  $\mathrm{GL}_{\mathbb{R}}(V)$ , which is closed by the following lemma.

**Lemma 5.4.** *Let  $V$  be a finite dimensional right  $\mathbb{D}$ -module, for  $\mathbb{D} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ . Let  $M \subseteq \mathrm{SL}_{\mathbb{D}}(V)$  and  $L \subseteq \mathbb{D}^*$  be closed subgroups. Then  $ML \subseteq \mathrm{GL}_{\mathbb{R}}(V)$  is a closed subgroup.*

*Proof.* We use the notation we have set up above. Put  $r = \dim_{\mathbb{D}}(V)$  and  $Z = \mathrm{SL}_{\mathbb{D}}(V) \cap U$ . Then  $|\det_{\mathbb{D}}(z)| = 1$  for all  $z \in Z$ . Hence we have a well-defined continuous homomorphism

$$\nu : \mathrm{SL}_{\mathbb{D}}(V)\mathbb{D}^* \longrightarrow \mathbb{R}_{>0}$$

given by

$$\nu(stu) = |\det_{\mathbb{D}}(st)| = |t|^r,$$

for  $s \in \mathrm{SL}_{\mathbb{D}}(V)$ ,  $t \in \mathbb{R}_{>0}$ , and  $u \in U$ . Now let  $(g_k)_{k \in \mathbb{N}}$  be a sequence in  $ML$  converging to an element  $g \in \mathrm{GL}_{\mathbb{R}}(V)$ . We express every  $g_k$  as a product  $g_k = s_k t_k u_k$ , with  $s_k \in M$ ,  $t_k \in \mathbb{R}_{>0}$  and  $u_k \in U$ . Then  $\nu(g) = \lim_k \nu(g_k) = \lim_k |t_k|^r$ . Passing to a subsequence, we may therefore assume that the bounded sequences  $t_k$  and  $u_k$  converge to elements  $t$  and  $u$ , respectively. Thus  $\lim_k t_k u_k = tu \in L$ . Hence

$$s = \lim_k s_k = \lim_k g_k u_k^{-1} t_k^{-1}$$

exists and is equal to  $gu^{-1}t^{-1}$ . Since  $M$  was assumed to be closed,  $s \in M$  and hence  $g = stu \in ML$ .  $\square$

**Theorem 5.5.** *Let  $V$  be a finite dimensional real vector space and let  $M \subseteq \mathrm{GL}(V)$  be a closed subgroup, with  $M/M^\circ$  finite. Assume that  $\mathrm{Lie}(M)$  is a direct sum of absolutely simple ideals and that  $M^\circ$  acts irreducibly on  $V$ . Let  $L \subseteq \mathrm{End}_M(V)^* = \mathbb{D}^*$  be a closed nontrivial subgroup. Let  $\rho$  denote the representation of  $ML$  on  $V$ . Then  $ML \subseteq \mathrm{GL}_{\mathbb{R}}(V)$  is closed and  $G = V \rtimes_{\rho} ML$  is rigid within every almost Polish class  $\mathcal{K}$  containing  $G$ .*

*Proof.* We view  $V$  as a right  $\mathbb{D}$ -module, for  $\mathbb{D} = \mathrm{End}_M(V)$ . The Lie algebra of  $M$  is perfect. Therefore  $M^\circ$  is contained in  $\mathrm{SL}_{\mathbb{D}}(V)$ . By Lemma 5.4, the group  $M^\circ L$  is closed in  $\mathrm{GL}_{\mathbb{R}}(V)$ . Since  $M/M^\circ$  is finite,  $ML$  is also closed in  $\mathrm{GL}_{\mathbb{R}}(V)$ .

We write the elements of  $G$  as pairs  $(u, m\ell)$ , with  $u \in V$ ,  $m \in M$  and  $\ell \in L$ . We are facing the problem that because of the presence of  $L$ , the Lie algebra of  $G$  need not be perfect, so additional work is required.

In any case,  $\mathrm{Lie}(V \rtimes_{\rho} M)$  is perfect by Lemma 5.2(3). Our first aim is to construct a good neighborhood basis of the identity of this subgroup of  $G$ . Let  $z \in L$  be a nontrivial element. Since  $z$  is not the identity in the division ring  $\mathbb{D}$ , it fixes no nonzero vector in  $V$ . Then

$$(u, m\ell)(0, z) = (u, m\ell z) \text{ and } (0, z)(u, m\ell) = (zu, zm\ell) = (zu, mz\ell),$$

hence

$$\mathrm{Cen}_G(z) = M \cdot \mathrm{Cen}_L(z).$$

Since  $M^\circ$  is linear and semisimple,  $\mathrm{Cen}(M^\circ)$  is finite, see [8, 38.5.(3)] or [16, Corollary 13.2.6]. By Proposition 4.5 we can find an element  $h$  and a finite set  $X$  in  $M$  such that

$$C = \{khk^{-1} \mid k \in \mathrm{Cen}_M(X)\}$$

is compact and spacious in  $M$ . Since  $L$  commutes with  $M$ , we may rewrite this set as

$$C = \{ghg^{-1} \mid g \in \mathrm{Cen}_G(X \cup \{z\})\}.$$

By Lemma 5.3, the set  $C$  is also spacious in  $V \rtimes_{\rho} M$ . Hence there exist elements  $g_1, \dots, g_r \in V \rtimes_{\rho} M$  such that  $D = g_1 C C^{-1} \cdots g_r C C^{-1}$  is a compact identity neighborhood in  $V \rtimes_{\rho} M$ . By Lemma 5.2(3) and Theorem 3.10, there exist 1-parameter groups  $c_1, \dots, c_m$  in  $V \rtimes_{\rho} M$  such that the sets

$$M_{1, \dots, t_m} = [c_1(t_1), D] \cdots [c_m(t_m), D],$$

for  $0 < t_i \leq 1$  and  $i = 1, \dots, m$ , form a neighborhood basis of the identity in  $V \rtimes_{\rho} M$ .

Now we construct a good neighborhood basis of the identity in  $L$ , using the sets  $M_{t_1, \dots, t_m} \subseteq V \rtimes_{\rho} M$ . Let  $X' \subseteq M$  be a finite set generating a dense subgroup. Since  $M/M^\circ$  is finite, such a set exists by Lemma 4.3. We have

$$L = \mathrm{Cen}_G(M) = \mathrm{Cen}_G(X'),$$

as is easily checked from the multiplication rule (\*\*). Recall that  $|\cdot|$  denotes the standard norm on the real division algebra  $\mathbb{D}$ . We fix also a norm  $\|\cdot\|$  on the right  $\mathbb{D}$ -module  $V$ ,

and a vector  $v_0 \in V$  with  $\|v_0\| = 1$ . We put

$$\begin{aligned} N_{t_1, \dots, t_m} &= \{\ell \in L \mid [(v_0, 1), (0, \ell)] \in M_{t_1, \dots, t_m}\} \\ &= \{\ell \in \text{Cen}_G(X') \mid [(v_0, 1), (0, \ell)] \in M_{t_1, \dots, t_m}\}. \end{aligned}$$

Since

$$[(v_0, 1), (0, \ell)] = (v_0, 1)(0, \ell)(-v_0, 1)(0, \ell^{-1}) = (v_0, \ell)(-v_0, \ell^{-1}) = (v_0(1 - \ell), 1)$$

and

$$\|v_0(\ell - 1)\| = |\ell - 1|,$$

we conclude that this family of sets is indeed a neighborhood basis of the identity in  $L$ , where  $0 < t_i \leq 1$  and  $i = 1, \dots, m$ .

We now combine the various pieces of this proof. The natural Lie group homomorphism

$$(V \rtimes_\rho M) \rtimes L \longrightarrow V \rtimes_\rho ML = G$$

is continuous and open. Given an identity neighborhood  $W \subseteq G$ , there exists therefore an identity neighborhood  $M_{t_1, \dots, t_m}$  in  $V \rtimes_\rho M$  and another identity neighborhood  $N_{t'_1, \dots, t'_m}$  in  $L$  such that the product  $M_{t_1, \dots, t_m} N_{t'_1, \dots, t'_m} \subseteq W$  is an identity neighborhood in  $G$ .

Suppose that  $K$  is another group in the class  $\mathcal{K}$  and that  $\varphi : K \longrightarrow G$  is an abstract surjective homomorphism whose kernel  $N$  is  $\mathcal{K}$ -analytic. Then both  $\varphi^{-1}(M_{t_1, \dots, t_m})$  and  $\varphi^{-1}(N_{t'_1, \dots, t'_m})$  are  $\mathcal{K}$ -analytic by Lemma 2.10, and so is  $\varphi^{-1}(M_{t_1, \dots, t_m} N_{t'_1, \dots, t'_m})$ . Hence  $\varphi$  is continuous and open by Theorem 2.9.  $\square$

The next result generalizes some of the results in [1].

**Theorem 5.6.** *The following Lie groups are rigid in every almost Polish class  $\mathcal{K}$  containing them:*

$$\begin{aligned} \mathbb{R}^n \rtimes_\rho \text{O}(n) & \quad n \geq 3, \\ \mathbb{R}^n \rtimes_\rho \text{SO}(n) & \quad n \geq 4, \\ \mathbb{C}^n \rtimes_\rho \text{SU}(n) & \quad n \geq 2, \\ \mathbb{H}^n \rtimes_\rho \text{Sp}(n) & \quad n \geq 1, \\ \mathbb{R}^{2n} \rtimes_\rho \text{Sp}_{2n}(\mathbb{R}) & \quad n \geq 1, \\ \mathbb{R}^n \rtimes_\rho \text{SL}_n(\mathbb{R}) & \quad n \geq 2, \\ \mathbb{R}^n \rtimes_\rho \text{GL}_n(\mathbb{R}) & \quad n \geq 2. \end{aligned}$$

*In all cases,  $\rho$  denotes the natural representation.*

Here  $\text{Sp}(n)$  denotes the quaternion unitary group acting on  $\mathbb{H}^n$ , and  $\text{Sp}_{2n}(\mathbb{R})$  denotes the group that leaves the standard symplectic form on  $\mathbb{R}^{2n}$  invariant. This list of applications of Theorem 5.5 to classical groups is by no means complete.

The reader should keep in mind, though, that

$$\mathbb{R}^3 \rtimes_{\text{Ad}} \text{SO}(3) = \text{SO}_3(\mathbb{R}[\delta])$$

is not rigid by Example 1.5. Likewise, the Lie group  $O_3(\mathbb{R}[\delta])$  is not rigid. Note, however, that for the natural representation  $\rho : O(3) \hookrightarrow GL_3(\mathbb{R})$  we have

$$\mathbb{R}^3 \rtimes_{\rho} O(3) \not\cong \text{Lie}(O(3)) \rtimes_{\text{Ad}} O(3) = O_3(\mathbb{R}[\delta]),$$

because  $-1 \in O(3)$  acts via  $\rho$  nontrivially on  $\mathbb{R}^3$ , whereas  $-1$  acts trivially on  $\text{Lie}(O(3))$  via  $\text{Ad}$ .

*Proof of Theorem 5.6.* We have only to consider the cases of the groups of  $\mathbb{R}^n \rtimes_{\rho} SO(n)$ , for  $n \geq 5$  odd, and  $\mathbb{R}^n \rtimes_{\rho} SL_n(\mathbb{R})$ , for  $n \geq 3$  odd. In all other cases, Theorem 5.5 applies directly.

Put  $H(n) = SO(n)$  or  $H(n) = SL_n(\mathbb{R})$  and  $G = \mathbb{R}^n \rtimes_{\rho} H(n)$ , where  $\rho : H(n) \hookrightarrow GL_n(\mathbb{R})$  is the natural representation. Suppose that  $n$  is odd. We decompose the matrices in  $H(n)$  into  $2 \times 2$  block matrices  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , with  $a$  of size  $(n-1) \times (n-1)$ . Put  $z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in H(n)$ . The  $G$ -centralizer of  $z$  consists of all pairs  $(w, g)$ , where  $w \in \mathbb{R}^n$  is a vector whose first  $n-1$  entries are 0, and  $g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ . Put

$$L = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in H(n) \right\}.$$

Since  $n$  is odd,  $d^{-1/n}$  exists in  $\mathbb{R}$ , and  $\det(d^{-1/n}a) = 1$ . The group  $H(n-1)$  injects into  $H(n)$  as the group of block diagonal matrices  $b = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$ . Every element in  $L$  is thus a product of a matrix in  $H(n-1)$  and a diagonal matrix of the form  $\begin{pmatrix} d^{-1/n} & 0 \\ 0 & d \end{pmatrix}$ . By our assumptions on  $n$ , the group  $H(n-1)$  has a semisimple Lie algebra whose simple ideals are absolutely simple. Note that here the assumption  $n \geq 5$  enters in the orthogonal case. Now we choose  $h$  and  $X$  in  $H(n-1)$  as in Proposition 4.5. Then

$$\begin{aligned} C &= \{ghg^{-1} \mid g \in \text{Cen}_{H(n-1)}(X)\} \\ &= \{ghg^{-1} \mid g \in \text{Cen}_L(X)\} \\ &= \{ghg^{-1} \mid g \in \text{Cen}_{H(n)}(\{z\} \cup X)\} \\ &= \{ghg^{-1} \mid g \in \text{Cen}_G(\{z\} \cup X)\}. \end{aligned}$$

The last equality follows since  $h$ , being an element of  $H(n-1)$ , fixes every vector  $w$  whose first  $n-1$  coordinates are 0. The set  $C$  is compact and contains a nonconstant path. Since  $\text{Lie}(H(n))$  is absolutely simple,  $C$  is spacious in  $H(n)$  by Proposition 4.4. By Lemma 5.3, the set  $C$  is spacious in  $G$ .

Suppose that  $K$  is a group in  $\mathcal{K}$  and that  $\varphi : K \rightarrow G$  is an abstract surjective homomorphism whose kernel  $N$  is  $\mathcal{K}$ -analytic. By Lemma 2.10, the set  $\varphi^{-1}(C)$  is  $\mathcal{K}$ -analytic. Hence  $G$  is rigid by Theorem 4.1.  $\square$

## 6. RIGIDITY OF FINITELY GENERATED PROFINITE GROUPS

We recall the definition of a verbal subgroup in a group  $G$ .



**6.1. Verbal subgroups.** Let  $G$  be a group and let  $F_s$  denote the free group on  $s$  generators  $x_1, \dots, x_s$ . By the universal property of  $F_s$ , every  $s$ -tuple  $\mathbf{g} = (g_1, \dots, g_s)$  of elements of  $G$  determines a unique homomorphism  $\varphi_{\mathbf{g}} : F_s \rightarrow G$ , with  $\varphi_{\mathbf{g}}(x_i) = g_i$ . Let  $w \in F_s$  be a word, i.e. an element in the free group. The *word map*

$$w(-) : G \times \dots \times G \rightarrow G$$

maps  $\mathbf{g}$  to  $\varphi_{\mathbf{g}}(w)$ . To put it differently, we express  $w$  as a word  $w = w(x_1, \dots, x_s)$  in the generators  $x_1, \dots, x_s$  and then we substitute  $g_i$  for  $x_i$  in  $w$ , for  $i = 1, \dots, s$ , and consider the resulting element  $w(g_1, \dots, g_s)$  in  $G$ . The image of the word map is the *verbal set*

$$G^w = \{\varphi_{\mathbf{g}}(w) \mid \mathbf{g} \in G \times \dots \times G\} = \{w(g_1, \dots, g_s) \mid g_1, \dots, g_s \in G\}.$$

The corresponding *verbal subgroup* is

$$w(G) = \langle G^w \rangle.$$

For example if  $s = 2$  and  $w = [x_1, x_2]$ , then  $G^w$  is the set of all commutators in  $G$  and  $w(G)$  is the commutator group of  $G$ .

A topological group  $G$  is called *topologically finitely generated* if there exists a finitely generated dense subgroup in  $G$ . If  $G$  is compact then  $G$  is said to have *finite generating rank* in the sense of [21, Definition 12.15]. Such groups are automatically second countable if they are even profinite by [21, Proposition 12.28] and so they are metrizable by [21, Corollary A4.19, p. 838].

Nikolov–Segal proved that in a topologically finitely generated profinite group  $G$  every abstract subgroup of finite index is open [33, Theorem 1.1]. In the course of their proof they show implicitly that every open subgroup of  $G$  contains a verbal open subgroup; this is worked out in detail in [9, Lemma 15]. Using this result, it is shown in [9, Theorem 16] that every topologically finitely generated profinite group  $G$  carries a *unique* Polish group topology. We generalize this below. First we need an auxiliary result.

**Lemma 6.2.** *Let  $\mathcal{K}$  be an almost Polish class, let  $G$  be a topological group in  $\mathcal{K}$ , with a normal (but not necessarily closed) subgroup  $N \trianglelefteq G$  which is  $\mathcal{K}$ -analytic. Put  $H = G/N$  (as an abstract group) and let  $\pi : G \rightarrow H$  denote the quotient map. Let  $w$  be a word in  $F_s$ . Then  $\pi^{-1}(H^w)$  is  $\mathcal{K}$ -analytic and  $\pi^{-1}(w(H))$  is almost open.*

*Proof.* We proceed similarly as in the proof of Lemma 2.10. The set

$$X = \pi^{-1}(H^w) = \{w(g_1, \dots, g_s)n \mid g_i \in G, n \in N\} = G^w N$$

is  $\mathcal{K}$ -analytic by Lemma 2.10(1), and so is  $X^{-1}$ . Again by Lemma 2.10(1), each term in the countable union

$$\pi^{-1}(w(H)) = \bigcup_{n \geq 1} (X X^{-1})^{\cdot n}$$

is  $\mathcal{K}$ -analytic. Since  $\mathcal{K}$  is almost Polish, the  $\mathcal{K}$ -analytic sets  $(X X^{-1})^{\cdot n}$  are almost open. Since the almost open sets form a  $\sigma$ -algebra,  $\pi^{-1}(w(H))$  is also almost open.  $\square$

We have the following generalization of [9, Theorem 16], which shows in particular that a topologically finitely generated profinite group is rigid within the classes  $\mathcal{L}^\sigma$ ,  $\mathcal{C}$  and  $\mathcal{P}$ . This is Theorem C from the introduction.

**Theorem 6.3.** *Let  $G$  be a topologically finitely generated profinite group. Let  $\mathcal{K}$  be an almost Polish class. If  $G$  is contained in  $\mathcal{K}$ , then  $G$  is rigid within  $\mathcal{K}$ .*

*Proof.* Suppose that  $1 \longrightarrow N \hookrightarrow K \xrightarrow{\varphi} G \longrightarrow 1$  is a short exact sequence of groups, where  $\varphi$  is an abstract group homomorphism, and that  $(K, N)$  is in  $\mathcal{K}_a$ . By [9, Lemma 15], every identity neighborhood  $U \subseteq G$  contains an open verbal subgroup  $w(G)$ , for some word  $w$  in some free group  $F_s$ . By Lemma 6.2, the preimage  $\varphi^{-1}(w(G))$  is almost open in  $K$ . Hence  $\varphi$  is continuous and open by Theorem 2.9.  $\square$

## 7. RIGIDITY OF COMPACT SEMISIMPLE GROUPS

A compact connected group is called *semisimple* if it coincides with its commutator group, see [21, Statements 9.4, 9.5, 9.6]. We recall the structure theorem for compact semisimple groups from [21, Theorem 9.19]. Many more structural results about such groups can be found in Chapter 9 in *loc.cit.* Let us call a Lie group *almost simple* if it is connected and if its Lie algebra is simple. We call a group homomorphism *central* if its kernel is contained in the center of its domain.

**Theorem 7.1.** [21, Theorem 9.19 and Theorem 9.2] *Let  $G$  be a compact connected semisimple group. Then there exists a family of compact almost simple Lie groups  $(S_i)_{i \in I}$  and a central continuous open surjective homomorphism*

$$\rho : \prod_{i \in I} S_i \longrightarrow G.$$

*Moreover, every element in  $G$  is a commutator.*

It turns out that for proving automatic continuity of abstract homomorphisms onto compact connected semisimple groups, we need only the abstract group-theoretic properties stated in Theorem 7.1. This abstraction allows us to prove a continuity result which applies also to a wide class of profinite groups. We set the stage as follows.

**7.2. Quasisimple and quasi-semisimple groups.** Extending widespread terminology from finite group theory, we call a nontrivial compact group  $S$  *quasisimple* if its abstract commutator group is dense and if  $S/\text{Cen}(S)$  is topologically simple (meaning that  $S/\text{Cen}(S)$  is nontrivial and has no nontrivial closed proper normal subgroups).

We call a compact group  $G$  *quasi-semisimple* if there exists a family of compact quasisimple groups  $(S_i)_{i \in I}$  and a continuous surjective central homomorphism

$$(***) \quad \prod_{i \in I} S_i \xrightarrow{\rho} G.$$

Hence every compact connected semisimple group is quasi-semisimple. We note that  $\rho$  is automatically open, either by the Open Mapping Theorem [14, II.5.29], [19], [40, 6.19], or alternatively by Theorem 2.9, applied to the almost Polish class  $\mathcal{C}$  of compact spaces. Thus every compact connected semisimple group is quasi-semisimple. But also profinite groups like  $\prod_{p \in \mathbb{P}} \mathrm{SL}_2(\mathbb{F}_p)$  are quasi-semisimple, where  $\mathbb{P}$  denotes the set of all primes and  $\mathbb{F}_p$  the field of  $p$  elements.

Let  $\rho$  be a homomorphism as in (\*\*\*) above. For a subset  $J \subseteq I$  we put  $S_J = \prod_{j \in J} S_j$ , with the convention that  $S_\emptyset = \{1\}$ . We view this group as a compact subgroup of  $S_I$ , and we put  $G_J = \rho(S_J)$ . Thus  $G_J$  is a compact quasi-semisimple subgroup of  $G = G_I$ . If  $I = J \sqcup K$  is a partition of  $I$ , then  $G_J$  and  $G_K$  commute, whence  $G_J \cap G_K \subseteq \mathrm{Cen}(G)$ , and  $G = G_J G_K$ .

The next proposition clarifies the structure of compact quasisimple groups. It depends heavily on several deep results.

**Proposition 7.3.** *Suppose that  $S$  is a compact quasisimple group. Then either  $S$  is a finite quasisimple group, or  $S$  is a compact almost simple Lie group. Every element in  $S$  is a product of (at most) 2 commutators.*

*Proof.* Put  $H = S/\mathrm{Cen}(S)$ . By [21, 9.90], the group  $H$  is simple as an abstract group, and  $H$  is a compact Lie group (first paragraph of the proof in *loc.cit.*). The identity component  $H^\circ$  is thus either trivial, or  $H^\circ = H$ . Let  $S' \subseteq S$  denote the abstract commutator group of  $S$ . Note that  $S'$  is not contained in the center of  $S$ , since otherwise  $S = \overline{S'}$  would be abelian, and then  $S'$  and hence  $S$  would be trivial. Therefore  $S'$  surjects onto the simple group  $H$ . In particular,  $H$  is perfect and thus nonabelian.

If  $H^\circ = \{1\}$ , then  $H$ , being a compact Lie group, is finite and Schur's Theorem [36, 10.1.4] implies that  $S$  is finite and perfect. By the deep result [30], every element in  $S$  is the product of at most 2 commutators.

If  $H^\circ = H$ , then  $H$  is in particular a compact almost simple Lie group. We put  $Z = \mathrm{Cen}(S)$  and we claim that  $S$  is connected. The image of  $S^\circ$  in  $H$  is a closed normal subgroup and hence either trivial, or it coincides with  $H$ . In the first case,  $S^\circ \subseteq Z$ . But then the compact totally disconnected group  $S/S^\circ$  maps onto  $H$  through an open homomorphism, contradicting the fact that  $H$  is connected. Hence  $S^\circ$  maps onto  $H$ , that is,  $S^\circ Z = S$ . But then every commutator of elements in  $S$  is contained in  $S^\circ$ , and therefore  $S = S^\circ$  is a compact connected semisimple group [21, 9.3 and 9.5]. From the Structure Theorem 7.1 and the fact that  $S/Z$  is almost simple, we conclude that  $S$  is an almost simple Lie group. By Gotô's Theorem [21, 9.2], every element in  $S$  is a commutator.  $\square$

**Corollary 7.4.** *In a quasi-semisimple group, every element is a product of 2 commutators.*

For the next two lemmas we assume that  $G$  is quasi-semisimple as in Definition 7.2, that  $(S_i)_{i \in I}$  is a family of compact quasisimple groups, and that  $\rho : \prod_{i \in I} S_i \rightarrow G$  is a central surjective continuous homomorphism. Note that  $\rho$  is open by the remark above.

**Lemma 7.5.** *If  $I = J \sqcup K$  is a partition of  $I$ , then*

$$\text{Cen}_G(G_K) = G_J \text{Cen}(G_K) \text{ and } G_J = \{[g_1, g_2][h_1, h_2] \mid g_1, g_2, h_1, h_2 \in \text{Cen}_G(G_K)\}.$$

*Proof.* Every element  $g \in G$  can be written as  $g = ab$ , with  $a \in G_J$  and  $b \in G_K$ . Let  $h \in G_K$ . Then  $abh = hab$  holds if and only if  $bh = hb$ . Hence the centralizer of  $G_K$  consists of all elements of the form  $ab$ , with  $a \in G_J$  and  $b \in \text{Cen}(G_K)$ . The commutator of two such elements is  $[a_1b_1, a_2b_2] = [a_1, a_2][b_1, b_2] = [a_1, a_2] \in G_J$ . Since  $G_J$  is quasi-semisimple, every element in  $G_J$  is a product of two commutators of elements of  $G_J$ .  $\square$

**Lemma 7.6.** *Let  $U \subseteq G$  be an identity neighborhood. Then there exists a finite subset  $J \subseteq I$ , and an identity neighborhood  $W$  in  $G_J$  such that  $WG_{I-J} \subseteq U$  is an identity neighborhood in  $G$ .*

*Proof.* By the definition of the product topology and from the continuity of  $\rho$ , we find a finite index set  $J \subseteq I$  and an identity neighborhood  $W' \subseteq S_J$  such that  $W'S_{I-J} \subseteq S_I$  is an identity neighborhood, with  $\rho(W'S_{I-J}) \subseteq U$ . Since  $\rho : S_J \rightarrow G_J$  is open, the set  $W = \rho(W')$  is an identity neighborhood in  $G_J$ , and similarly  $\rho(W'S_{I-J}) = WG_{I-J}$  is an identity neighborhood in  $G$ .  $\square$

The next result implies Theorem B in the introduction. Note, however, that Theorem 7.7 applies also to profinite quasi-semisimple groups, which need not be topologically finitely generated, separable, or metrizable.

**Theorem 7.7.** *Let  $G$  be a compact quasi-semisimple group, and let  $\mathcal{K}$  be an almost Polish class. If  $G$  belongs to  $\mathcal{K}$ , then  $G$  is rigid within  $\mathcal{K}$ .*

*Proof.* Suppose that

$$1 \longrightarrow N \hookrightarrow K \xrightarrow{\varphi} G \longrightarrow 1$$

is a short exact sequence of groups, where  $\varphi$  is an abstract group homomorphism, and that  $(K, N) \in \mathcal{K}_a$ . We use the notation that we have set up above. Let  $U \subseteq G$  be an identity neighborhood. We choose  $J \subseteq I$  finite and an identity neighborhood  $W \subseteq G_J$  as in Lemma 7.6 above, such that  $WG_{I-J} \subseteq U$ . Since  $G_J$  is a compact Lie group, there exists a finite subset  $X \subseteq G_J$  which generates a dense subgroup in  $G_J$ , see Lemma 4.3. By Lemma 7.5,

$$G_{I-J} = \{[g_1, g_2][h_1, h_2] \mid g_1, g_2, h_1, h_2 \in \text{Cen}_G(X)\}.$$

Then  $\varphi^{-1}(G_{I-J})$  is  $\mathcal{K}$ -analytic by Lemma 2.10. The Lie algebra of  $G_J$  is semisimple. Hence we may choose elements  $a_1, \dots, a_r$  in the compact Lie group  $G_J$  as in Theorem 3.10, such that

$$V = [a_1, G_J] \cdots [a_r, G_J] = [a_1, G] \cdots [a_r, G] \subseteq W$$

is a compact identity neighborhood in  $G_J$ . Then  $VG_{I-J} \subseteq WG_{I-J} \subseteq U$  is a compact identity neighborhood, and  $\varphi^{-1}(VG_{I-J})$  is  $\mathcal{K}$ -analytic by Lemma 2.10. The claim follows now from Theorem 2.9.  $\square$

## 8. THE PROOF OF THEOREM D

In this last section we consider abstract homomorphisms  $\psi : G \longrightarrow H$ , where  $H$  is a topological group and  $G$  is a Lie group whose Lie algebra is perfect. The following result is Theorem D from the introduction. It generalizes some of the the main results in [37].

**Theorem 8.1.** *Let  $G$  be a Lie group whose Lie algebra is perfect. Let  $H$  be a topological group, and let  $\psi : G \longrightarrow H$  be an abstract homomorphism. If there exists a compact spacious set  $C \subseteq G$  whose image  $\psi(C)$  has compact closure, then  $\psi$  is continuous.*

*Proof.* We follow the strategy of [21, Theorem 5.64]. Let  $D = g_1 C C^{-1} \cdots g_r C C^{-1} \subseteq G$  be a compact identity neighborhood. It follows that  $\psi(D) \subseteq H$  has compact closure  $E = \overline{\psi(D)}$ . Put  $n = \dim(G)$ . Let  $U \subseteq H$  be an arbitrary identity neighborhood. By Wallace's Lemma 3.9, there exists an identity neighborhood  $W \subseteq H$  such that  $[h_1, E] \cdots [h_n, E] \subseteq U$  for all  $h_1, \dots, h_n \in W$ .

We choose 1-parameter groups  $c_1, \dots, c_n$  in  $G$  as in Theorem 3.10. We claim that we can find numbers  $t_i$  with  $0 < |t_i| \leq 1$  such that  $h_i = \psi(c_i(t_i)) \in W$ . Once we manage to do this, we have  $\psi([c_1(t_1), D] \cdots [c_n(t_n), D]) \subseteq [h_1, E] \cdots [h_n, E] \subseteq U$  and the continuity of  $\psi$  at the identity follows, because  $[c_1(t_1), D] \cdots [c_n(t_n), D]$  is by Theorem 3.10 an identity neighborhood. Then the global continuity of  $\psi$  follows [3, III. Proposition 23].

Fix  $i$  and put  $\tau = \psi \circ c_i : \mathbb{R} \longrightarrow H$ . There exists a number  $0 < s < 1$  such that  $c_i([0, s]) \subseteq D$ . The interval  $P = [0, s]$  generates  $(\mathbb{R}, +)$  as a group, hence  $Q = \tau(P) \subseteq E$  generates the group  $A = \tau(\mathbb{R})$ . Since  $A$  is divisible, we have either  $A = \{1\} = Q$ , or  $Q$  is infinite, because the only finitely generated divisible abelian group is the trivial group.

If  $A = \{1\}$  put  $t_i = 1$ .

Otherwise,  $Q$  is infinite, hence  $Q \subseteq E$  has an accumulation point  $h \in E$ , because  $E$  is compact. Let  $V \subseteq H$  be an open identity neighborhood such that  $VV^{-1} \subseteq W$ . Then  $Vh$  contains infinitely many elements of  $Q$ . We choose numbers  $a, b$  with  $0 \leq a < b \leq s$  such that  $\tau(a), \tau(b) \in Vh$ . Then  $0 < b - a \leq 1$  and  $\tau(b - a) = \tau(b)\tau(a)^{-1} \in Vh(Vh)^{-1} = VV^{-1} \subseteq W$ . Hence we may put  $t_i = b - a$ .  $\square$

**Corollary 8.2.** [21, Theorem 5.64] *Let  $G$  be a Lie group whose Lie algebra is perfect, let  $H$  be a compact group and let  $\psi : G \longrightarrow H$  be an abstract homomorphism. Then  $\psi$  is continuous.*

**Corollary 8.3.** *Let  $G$  be a Lie group whose Lie algebra is simple, let  $H$  be a topological group and let  $\psi : G \longrightarrow H$  be an abstract homomorphism. If there exists a nonconstant path  $f : [0, 1] \longrightarrow G$  such that  $\psi(f([0, 1])) \subseteq H$  has compact closure, then  $\psi$  is continuous.*

## 9. AN ERRATUM AND A COMMENT

We take the opportunity to correct a mistake which occurred in [28]. In Theorem 7 and Corollary 8 in *loc.cit.*, the hypothesis has to be added that the kernel of the homomorphism  $\varphi$  is  $\sigma$ -compact.<sup>4</sup> The same assumption has to be added in Proposition 37 and in Proposition 41 in [4]. Example 1.3 in the present article shows that this hypothesis on the kernel cannot be omitted. Of course this hypothesis on the kernel is satisfied if  $\varphi$  is bijective. Theorem 11 in [28] is therefore not affected by the mistake. However, this result is superseded by Theorem 4.6 in the present article, which is more general. There is also a small misprint in the statement of Theorem 3 in [28], which should read ‘*if  $G$  is  $\sigma$ -compact, then...*’. This has no consequences, and was kindly pointed out by Ruppert McCallum.

We finally remark that the first paragraphs in [38] might give the erroneous impression that [28, Theorem 18] is incorrect, and that [28, Theorem 11] is implied by the author’s earlier work on locally bounded homomorphisms.

## REFERENCES

- [1] W. M. Al-Tameemi and R. R. Kallman, The natural semidirect product  $\mathbb{R}^n \rtimes G(n)$  is algebraically determined, *Topology Appl.* **199** (2016), 70–83. MR3442596
- [2] A. Borel and J. Tits, Homomorphismes “abstraites” de groupes algébriques simples, *Ann. of Math.* (2) **97** (1973), 499–571. MR0316587 (47 #5134)
- [3] N. Bourbaki, *General topology. Chapters 1–4*, translated from the French, reprint of the 1966 edition, *Elements of Mathematics* (Berlin), Springer, Berlin, 1989. MR0979294  
N. Bourbaki, *General topology. Chapters 5–10*, translated from the French, reprint of the 1966 edition, *Elements of Mathematics* (Berlin), Springer, Berlin, 1989. MR0979295
- [4] O. Braun, *Uniqueness of topologies on compact connected groups*. Diploma Thesis, Univ. Münster 2016.
- [5] É. Cartan, Sur les représentations linéaires des groupes clos, *Comment. Math. Helv.* **2** (1930), no. 1, 269–283. MR1509418
- [6] J. Dugundji, *Topology*, Allyn & Bacon, Boston, Mass., 1966. MR0193606
- [7] H. Freudenthal, Die Topologie der Lieschen Gruppen als algebraisches Phänomen. I, *Ann. of Math.* (2) **42** (1941), 1051–1074. MR0005740
- [8] H. Freudenthal and H. de Vries, *Linear Lie groups*, *Pure and Applied Mathematics*, Vol. 35, Academic Press, New York, 1969. MR0260926
- [9] P. Gartside and B. Pejić, Uniqueness of Polish group topology, *Topology Appl.* **155** (2008), no. 9, 992–999. MR2401209
- [10] V. M. Gluškov, Structure of locally bicomact groups and Hilbert’s fifth problem, *Uspehi Mat. Nauk* (N.S.) **12** (1957), no. 2 (74), 3–41. MR0101892 (21 #698)  
Translated as:  
V. M. Gluškov, The structure of locally compact groups and Hilbert’s fifth problem, *Amer. Math. Soc. Transl.* (2) **15** (1960), 55–93. MR0114872 (22 #5690)

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<sup>4</sup> The mistake occurs on p. 2625 where it is implicitly assumed that the preimage of a conjugacy class under a homomorphism is again a conjugacy class, which need not be the case.

- [11] M. Goto, On an arcwise connected subgroup of a Lie group, *Proc. Amer. Math. Soc.* **20** (1969), 157–162. MR0233923 (38 #2244)
- [12] H. Gündoğan, The component group of the automorphism group of a simple Lie algebra and the splitting of the corresponding short exact sequence, *J. Lie Theory* **20** (2010), no. 4, 709–737. MR2778233
- [13] S. Hernández, K. H. Hofmann and S. A. Morris, Nonmeasurable subgroups of compact groups, *J. Group Theory* **19** (2016), no. 1, 179–189. MR3441133
- [14] E. Hewitt and K. A. Ross, *Abstract harmonic analysis. Vol. I*, Second edition, Springer, Berlin, 1979. MR0551496
- [15] J. Hilgert, K. H. Hofmann and J. D. Lawson, *Lie groups, convex cones, and semigroups*, Oxford Mathematical Monographs, Oxford Univ. Press, New York, 1989. MR1032761
- [16] J. Hilgert and K.-H. Neeb, *Structure and geometry of Lie groups*, Springer Monographs in Mathematics, Springer, New York, 2012. MR3025417
- [17] K. H. Hofmann and L. Kramer, Transitive actions of locally compact groups on locally contractible spaces, *J. Reine Angew. Math.* **702** (2015), 227–243. MR3341471  
Erratum, *J. Reine Angew. Math.* **702** (2015), 245–246.
- [18] K. H. Hofmann and S. A. Morris, Transitive actions of compact groups and topological dimension, *J. Algebra* **234** (2000), no. 2, 454–479. MR1801101 (2002a:22006)
- [19] K. H. Hofmann and S. A. Morris, Open mapping theorem for topological groups, *Topology Proc.* **31** (2007), no. 2, 533–551. MR2476628
- [20] K. H. Hofmann and S. A. Morris, The structure of almost connected pro-Lie groups, *J. of Lie Theory* **21** (2011), 341–383. MR2828721
- [21] K. H. Hofmann and S. A. Morris, *The structure of compact groups*, third edition, revised and augmented., De Gruyter Studies in Mathematics, 25, de Gruyter, Berlin, 2013. MR3114697
- [22] K. H. Hofmann and S. A. Morris, Pro-Lie Groups: A Survey with Open Problems, *Axioms* **4** (2015), 294–312.
- [23] K. Iwasawa, On some types of topological groups, *Ann. of Math. (2)* **50** (1949), 507–558. MR0029911 (10,679a)
- [24] R. R. Kallman, The topology of compact simple Lie groups is essentially unique, *Advances in Math.* **12** (1974), 416–417. MR0357677
- [25] R. R. Kallman, A uniqueness result for a class of compact connected groups, in *Conference in modern analysis and probability (New Haven, Conn., 1982)*, 207–212, *Contemp. Math.*, 26, Amer. Math. Soc., Providence, RI. MR0737401
- [26] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, 156, Springer, New York, 1995. MR1321597
- [27] J. O. Kiltinen, On the number of field topologies on an infinite field, *Proc. Amer. Math. Soc.* **40** (1973), 30–36. MR0318118
- [28] L. Kramer, The topology of a semisimple Lie group is essentially unique, *Adv. Math.* **228** (2011), no. 5, 2623–2633. MR2838051
- [29] K. Kuratowski, *Topology. Vol. I*, New edition, revised and augmented. Translated from the French by J. Jaworowski, Academic Press, New York, 1966. MR0217751
- [30] M. W. Liebeck et al., Commutators in finite quasisimple groups, *Bull. Lond. Math. Soc.* **43** (2011), no. 6, 1079–1092. MR2861530
- [31] D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience Publishers, New York, 1955. MR0073104
- [32] S. Murakami, On the automorphisms of a real semi-simple Lie algebra, *J. Math. Soc. Japan* **4** (1952), 103–133. MR0051829

- [33] N. Nikolov and D. Segal, On finitely generated profinite groups. I. Strong completeness and uniform bounds, *Ann. of Math.* (2) **165** (2007), no. 1, 171–238. MR2276769
- [34] J. C. Oxtoby, *Measure and category*, second edition, Graduate Texts in Mathematics, 2, Springer, New York, 1980. MR0584443
- [35] B. J. Pettis, On continuity and openness of homomorphisms in topological groups, *Ann. of Math.* (2) **52** (1950), 293–308. MR0038358
- [36] D. J. S. Robinson, *A course in the theory of groups*, second edition, Graduate Texts in Mathematics, 80, Springer, New York, 1996. MR1357169
- [37] A. I. Shtern, Van der Waerden continuity theorem for semisimple Lie groups, *Russ. J. Math. Phys.* **13** (2006), no. 2, 210–223. MR2262825
- [38] A. I. Shtern, Bounded structure and continuity for homomorphisms of perfect connected locally compact groups, *Proc. Jangjeon Math. Soc.* **15** (2012), no. 3, 235–240. MR2978426
- [39] T. E. Stewart, Uniqueness of the topology in certain compact groups, *Trans. Amer. Math. Soc.* **97** (1960), 487–494. MR0126500
- [40] M. Stroppel, *Locally compact groups*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2006. MR2226087
- [41] J. Tits, Homomorphismes “abstraits” de groupes de Lie, in *Symposia Mathematica, Vol. XIII (Congresso di Gruppi e loro Rappresentazioni, INDAM, Rome, 1972)*, 479–499, Academic Press, London. MR0379749
- [42] B. L. van der Waerden, Stetigkeitssätze für halbeinfache Liesche Gruppen, *Math. Z.* **36** (1933), no. 1, 780–786. MR1545369
- [43] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, corrected reprint of the 1971 edition, Graduate Texts in Mathematics, 94, Springer, New York, 1983. MR0722297
- [44] G. Warner, *Harmonic analysis on semi-simple Lie groups. I*, Springer, New York, 1972. MR0498999

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